

# FRAMED HITCHIN PAIRS

ALEXANDER H.W. SCHMITT

**ABSTRACT.** We provide a construction of the moduli spaces of framed Hitchin pairs and their master spaces. These objects have come to interest as algebraic versions of solutions of certain coupled vortex equations. Our method unifies and generalizes constructions of several similar moduli spaces.

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## INTRODUCTION

The theory of stable vector bundles on complex projective manifolds has two completely different aspects: The algebro-geometric part of the construction of their moduli spaces by GIT, on the one hand, and, on the other hand, the theory of Hermitian-Einstein bundles which is differential geometry. The two theories are related by the famous Kobayashi-Hitchin correspondence. Using this, one could compute Donaldson invariants with the help of algebraic geometry. Now, Kobayashi-Hitchin type correspondences occur in many other places, e.g., for Bradlow pairs, Higgs pairs, and oriented pairs. Thus, it is desirable to have algebraic moduli spaces for the respective objects. For the above examples, the moduli spaces of the corresponding stable objects in algebraic geometry have been constructed (see, e.g., [11], [1], [7], [12], [9], [8]).

In this paper, we study framed Hitchin pairs and oriented framed Hitchin pairs from the algebro-geometric viewpoint. Let  $X$  be a smooth projective curve and fix line bundles  $L, M$ , and a vector bundle  $H$  on  $X$ . Then a framed Hitchin pair consists of a vector bundle  $E$  with  $\det(E) \cong M$ , a complex number  $\varepsilon$ , a twisted endomorphism  $\varphi: E \rightarrow E \otimes L$ , and a framing  $\psi: E \rightarrow H$ , and an oriented framed Hitchin pair consists of  $(E, \varepsilon, \varphi, \psi)$  as before, and an orientation  $\delta: \det(E) \rightarrow M$ . The motivation to study these objects comes from non-abelian Seiberg-Witten theory as explained in the recent thesis of M.-S. Stupariu [10]. He starts with the  $U(2)$ - and  $PU(2)$ -monopole equations on a hermitian rank two vector bundle on a Kähler surface and then applies the method of dimensional reduction to get new equations in complex dimension one. These are certain vortex-type equations coupled with Higgs fields. In the algebro-geometric setting, the  $U(2)$ - and  $PU(2)$ -monopole equations correspond to Bradlow pairs and oriented pairs, respectively, and — as in Hitchin's work — the process of dimensional reduction has the effect of "adding" a trace-free twisted endomorphism  $\varphi: E \rightarrow E \otimes K_X$  (see [10], p.29ff). In the spirit of the theory of complex vector bundles, one must now generalize the concept of Einstein-metrics to the respective differential-geometric objects, introduce a suitable stability concept for (oriented) framed Hitchin pairs, and then relate both sides by a Kobayashi-Hitchin correspondence. For framed Hitchin pairs, this was carried out by Lin [4]. The corresponding results for oriented framed Hitchin pairs are the content of Stupariu's thesis which also contains a discussion of Lin's results.

It is therefore important to construct the algebraic models for the moduli spaces of stable (oriented) framed Hitchin pairs together with their Gieseker compactifications. This will be the main concern of our note. We will define a very general notion of framed Hitchin pairs over arbitrary base manifolds, explain the correct notions of semistability, and carry

out a construction of the moduli spaces, using Geometric Invariant Theory, restricting ourselves to the case of curves in the case of oriented framed Hitchin pairs. In contrast to other constructions of very similar moduli spaces ([7], [12]), for framed Hitchin pairs, we will follow a Simpson-type construction, generalizing the one in [9]. The advantage of this approach is twofold. First, one immediately obtains projective moduli spaces, and second, the symmetricity condition that the induced homomorphism  $\wedge^2 \mathcal{G} \otimes \mathcal{E} \longrightarrow \mathcal{E}$  be zero, usually appearing in this context, can be suppressed.

## PRELIMINARIES

We work over  $\mathbb{C}$ . Here is a list of data which have to be fixed. We will refer to this list without further notice.

- $X$ , a smooth, projective scheme over the complex numbers,
- an ample sheaf  $\mathcal{O}_X(1)$ ,
- a Hilbert polynomial  $P$ ,  $r$  and  $d$  will denote the rank and the degree w.r.t.  $\mathcal{O}_X(1)$  which this polynomial determines,  $\mu := d/r$ ,
- a locally free sheaf  $\mathcal{G}$  on  $X$ ,
- a torsion free sheaf  $\mathcal{H}$  on  $X$ ,
- a Poincaré line bundle  $\mathcal{N}$  on  $\text{Pic } X \times X$ .

For any coherent sheaf  $\mathcal{E}$  on  $X$ ,  $P_{\mathcal{E}}$  and  $P(\mathcal{E})$  stand for its Hilbert polynomial w.r.t.  $\mathcal{O}_X(1)$ . For any scheme  $S$  and any  $S$ -flat coherent sheaf  $\mathfrak{E}_S$  on  $S \times X$ , there exists a morphism  $d(\mathfrak{E}_S): S \longrightarrow \text{Pic } X$ , associated to the line bundle  $\det(\mathfrak{E}_S)$  on  $S \times X$ . We write  $\mathcal{N}[\mathfrak{E}_S] := (d(\mathfrak{E}_S) \times \text{id}_X)^* \mathcal{N}$ .

A UNIVERSAL CONSTRUCTION. Remember the following standard construction which will be used frequently in our note:

PROPOSITION 0.1. *Let  $T$  be a noetherian scheme,  $\mathfrak{E}_T^1$  and  $\mathfrak{E}_T^2$   $T$ -flat coherent sheaves on  $T \times X$ , and  $\phi_T: \mathfrak{E}_T^1 \longrightarrow \mathfrak{E}_T^2$  a homomorphism. Then there is a closed subscheme  $\mathfrak{V} \subset T$  whose closed points are those  $t \in T$  for which  $\phi_{T|_{\{t\} \times X}} \equiv 0$ .*

THE COMMUTATION PRINCIPLE. Let us recall two results from [8] which will be used in the second part of our paper.

THEOREM 0.2. *Let  $G$  be a reductive group without characters, and suppose we are given a  $(\mathbb{C}^* \times G)$ -action on the projective scheme  $\mathfrak{X}$  which is linearized in, say,  $\mathfrak{M}$ . Then the following conditions are equivalent:*

1. *The point  $x$  is  $G$ -semistable w.r.t. the given linearization.*
2. *There exists a linearization  $l$  of the  $\mathbb{C}^*$ -action in some power  $\mathfrak{M}^{\otimes m}$  such that  $x$  is  $l$ -semistable, and the image of  $x$  in  $\mathfrak{X} //_l \mathbb{C}^*$  is a  $G$ -semistable point w.r.t. the induced linearization.*

*Proof.* [8], Thm.1.4.1. & Rem.1.1.1.  $\square$

Remark 0.3. Note that the same statement holds for polystable points but not for stable points.

PROPOSITION 0.4. *Let  $\mathbb{C}^*$  act on a vector space  $W_1 \oplus W_2$  with weights  $e_1$  and  $e_2$ ,  $e_1 < e_2$ . Then, for the linearization of the  $\mathbb{C}^*$ -action given by  $k \in \mathbb{Z}_{>0}$ ,  $e \in \mathbb{Z}$ , with  $e_1 < e/k < e_2$ , of the resulting  $\mathbb{C}^*$ -action on  $\mathbb{P}(W_1^\vee \oplus W_2^\vee)$  in  $\mathcal{O}(k)$ , the quotient is  $\mathbb{P}(W_1^\vee) \times \mathbb{P}(W_2^\vee)$ , and  $\mathcal{O}(k(e_2 - e_1))$  descends to  $\mathcal{O}(ke_2 - e, -ke_1 + e)$ . Furthermore, the linearizations with  $e/k = e_i$  yield as quotients  $\mathbb{P}(W_i^\vee)$ ,  $i = 1, 2$ ,  $\mathcal{O}(k(e_2 - e_1))$  descending to  $\mathcal{O}(k(e_2 - e_1))$ .*

*Proof.* [8], Example 1.2.5.  $\square$

THE PROBLEM OF NON-COMMUTING MATRICES. Denote by  $M_r$  the vector space of complex  $(r \times r)$ -matrices, and by  $\mathrm{SL}_r \subset M_r$  the special linear group. We are interested in the right action of  $\mathrm{SL}_r$  on  $W := M_r^{\oplus u}$  by conjugation, i.e.,  $(m_1, \dots, m_u) \cdot g = (g^{-1}m_1g, \dots, g^{-1}m_ug)$ , for  $(m_1, \dots, m_u) \in W$  and  $g \in \mathrm{SL}_r$ . First of all, it is very easy to describe the (semi)stable points:

LEMMA 0.5. *A point  $\underline{m} := (m_1, \dots, m_u)$  in  $W$  is unstable if and only if the  $m_i$ 's can be simultaneously triangularized, and  $\underline{m}$  fails to be semistable, if, in addition, all the  $m_i$ 's are nilpotent.*

Remark 0.6. We can formulate the condition of being a nullform in another way. For this, we think of  $\underline{m}$  as a linear map from  $\mathbb{C}^r$  to  $\mathbb{C}^r \otimes \mathbb{C}^u$ . Then,  $\underline{m}$  is a nullform if and only if  $(\underline{m} \otimes \mathrm{id}_{\mathbb{C}^u \otimes \mathbb{C}^{r-1}}) \circ \dots \circ \underline{m} = 0$ . When the latter happens, we call  $\underline{m}$  nilpotent.

For us, it will be important to know the ring of invariants  $\mathbb{C}[W]^{\mathrm{SL}_r} = \mathbb{C}[W]^{\mathrm{GL}_r}$ . From a technical point of view, the above problem is just a matrix problem associated with a quiver, namely with the one consisting of one vertex and  $u$  arrows, connecting the vertex to itself. Coordinate rings of general quiver varieties have been explicitly determined by Le Bruyn and Procesi [3], and Lusztig [5]. First, let us define some invariants. For this, let  $\mathbb{F}_u$  be the free group in  $u$  generators  $x_1, \dots, x_u$ . We think of the elements of  $\mathbb{F}_u$  as words in  $u$  letters. An element  $\omega \in \mathbb{F}_u$  and an element  $(m_1, \dots, m_u) \in M_r^u$  define a matrix  $m_\omega$  which is obtained by substituting  $m_i$  for the indeterminate  $x_i$ ,  $i = 1, \dots, u$ , and we associate to  $\omega$  the invariant  $T_\omega$  which assigns to  $(m_1, \dots, m_u)$  the Trace of  $m_\omega$ . Theorem 1 of [3] can be stated as follows in our context.

THEOREM 0.7. *The algebra  $\mathbb{C}[W]^{\mathrm{SL}_r}$  is generated by the elements  $T_\omega$  belonging to words  $\omega$  of length at most  $r^2$ .*

Remark 0.8. As a dimension count shows, the number of invariants  $s$  is much bigger than the dimension of the quotient, whence a lot of relations must hold. These follow all from the Cayley-Hamilton theorem.

Example 0.9 (The case  $r = 2 = u$ ). For  $(m_1, m_2) \in M_2 \oplus M_2 =: W$ , we define the invariants

$$\begin{aligned} \tilde{T}_1(m_1, m_2) &:= \mathrm{Trace}(m_1), & \tilde{T}_2(m_1, m_2) &:= \det(m_1), \\ \tilde{T}_3(m_1, m_2) &:= \mathrm{Trace}(m_2), & \tilde{T}_4(m_1, m_2) &:= \det(m_2), \\ \text{and } \tilde{T}_5(m_1, m_2) &:= \mathrm{Trace}(m_1 m_2). \end{aligned}$$

One can then verify that  $\mathbb{C}[W]^{\mathrm{SL}_2} = \mathbb{C}[\tilde{T}_1, \dots, \tilde{T}_5]$  and that this ring is isomorphic to the polynomial ring in five variables.

Let  $s$  be the number of (non-empty) words of length at most  $r^2$  in  $u$  letters,  $T_1, \dots, T_s$  be the generating invariants from Thm. 0.7, and  $\Xi := \mathbb{A}^s$ . Thus, we can associate to any  $\underline{m}$  the element  $\xi(\underline{m}) := (T_1(\underline{m}), \dots, T_s(\underline{m})) \in \Xi$  which we call *the characteristic vector* of  $\underline{m}$ . This is the replacement for the characteristic polynomials in the case of commuting matrices. The results of this paragraph can be elegantly stated as

*$\underline{m}$  is not nilpotent if and only if its characteristic vector is non-zero.*

## 1. FRAMED HITCHIN PAIRS

DEFINITIONS. We will now introduce framed Hitchin pairs. They form a large class of objects which comprises all the objects studied in [1], [7], [12], and [9].

*Framed Hitchin pairs.* A framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  is a quadruple  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  composed of the following ingredients

- a torsion free coherent sheaf  $\mathcal{E}$  with  $P_{\mathcal{E}} = P$ ,
- a complex number  $\varepsilon$ ,
- a twisted endomorphism  $\varphi: \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{G}$ ,
- a non-zero framing  $\psi: \mathcal{E} \rightarrow \mathcal{H}$ ,

such that  $\varphi$  is not nilpotent, i.e.,  $(\varphi \otimes \text{id}_{\mathcal{G}^{\otimes i-1}}) \circ \dots \circ \varphi \neq 0$ , for all  $i \in \mathbb{N}$ , when  $\varepsilon = 0$ . To make the condition of nilpotency a bit more transparent, we state the following

LEMMA 1.1. *Let  $\mathcal{E}$  be a torsion free sheaf of rank  $r$  and  $\varphi: \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{G}$  a twisted endomorphism. Then  $\varphi$  is nilpotent in the above sense if and only if  $(\varphi \otimes \text{id}_{\mathcal{G}^{\otimes r-1}}) \circ \dots \circ \varphi = 0$ .*

*Proof.* One direction is trivial. Suppose  $\varphi$  is nilpotent and set  $\mathcal{F}_i := \ker((\varphi \otimes \text{id}_{\mathcal{G}^{\otimes i-1}}) \circ \dots \circ \varphi)$ ,  $i \in \mathbb{N}$ , so that we get a filtration  $0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_s = \mathcal{E}$ . Now, the  $\mathcal{F}_i$ 's are saturated subsheaves of  $\mathcal{E}$ . Hence, all the inclusions above are either equalities or the rank jumps by one. Thus, the assertion follows from the obvious fact  $\varphi(\mathcal{F}_i) \subset \mathcal{F}_{i-1} \otimes \mathcal{G}$ .  $\square$

The equivalence relation  $\sim$  on framed Hitchin pairs is the one generated by

$$\begin{aligned} (\mathcal{E}, \varepsilon, \varphi, \psi) &\sim (\mathcal{E}', \varepsilon, (\rho \otimes \text{id}_{\mathcal{G}}) \circ \varphi \circ \rho^{-1}, \psi \circ \rho^{-1}), \quad \rho: \mathcal{E} \rightarrow \mathcal{E}' \text{ iso.}, \\ (\mathcal{E}, \varepsilon, \varphi, \psi) &\sim (\mathcal{E}, z \cdot \varepsilon, z \cdot \varphi, \psi), \quad z \in \mathbb{C}^*. \end{aligned}$$

Remark 1.2. Applying the above definition to the automorphism  $\lambda \cdot \text{id}$ ,  $\lambda \in \mathbb{C}^*$ , one sees that  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is always equivalent to  $(\mathcal{E}, \varepsilon, \varphi, \lambda \cdot \psi)$ .

A family of framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H})$  parametrized by the (noetherian) scheme  $S$  is a quintuple  $(\mathcal{E}_S, \varepsilon_S, \varphi_S, \psi_S, \mathcal{N}_S)$ . Here,  $\mathcal{E}_S$  is an  $S$ -flat family of torsion free coherent sheaves on  $S \times X$  with Hilbert polynomial  $P$ ,  $\mathcal{N}_S$  is a line bundle on  $S$ ,  $\varepsilon_S \in H^0(\mathcal{N}_S)$ ,  $\varphi_S: \mathcal{E}_S \rightarrow \mathcal{E}_S \otimes \pi_S^* \mathcal{N}_S \otimes \pi_X^* \mathcal{G}$  is a twisted endomorphism, and  $\psi_S: \mathcal{E}_S \rightarrow \pi_X^* \mathcal{H}$  is a framing. We say that the family  $(\mathcal{E}_S, \varepsilon_S, \varphi_S, \psi_S, \mathcal{N}_S)$  is *equivalent* to the family  $(\mathcal{E}'_S, \varepsilon'_S, \varphi'_S, \psi'_S, \mathcal{N}'_S)$ , if we find isomorphisms  $\rho_S: \mathcal{E}_S \rightarrow \mathcal{E}'_S$  and  $\eta_S: \mathcal{N}_S \rightarrow \mathcal{N}'_S$ , such that

$$\varepsilon'_S = \eta_S \circ \varepsilon_S, \quad \varphi'_S = (\rho_S \otimes \pi_S^* \eta_S \otimes \text{id}_{\pi_X^* \mathcal{G}}) \circ \varphi_S \circ \rho_S^{-1}, \quad \psi'_S = \psi_S \circ \rho_S^{-1}.$$

*Symmetric framed Hitchin pairs and characteristic polynomials.* Let  $\mathcal{E}$  be a torsion free coherent sheaf and  $\varphi: \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{G}$  be a twisted endomorphism. We call  $\varphi$  *symmetric*, if the induced homomorphism  $\mathcal{E} \otimes T^*(\mathcal{G}^\vee) \rightarrow \mathcal{E}$  factors through  $\mathcal{E} \otimes S^*(\mathcal{G}^\vee)$ .

Remark 1.3. Since the symmetric algebra  $S^*(\mathcal{G}^\vee)$  is generated over  $\mathcal{O}_X$  by  $\mathcal{G}^\vee$  it is sufficient to have that the map from  $\mathcal{E} \otimes \mathcal{G}^\vee \otimes \mathcal{G}^\vee \rightarrow \mathcal{E}$  vanishes on  $\mathcal{E} \otimes (\wedge^2 \mathcal{G}^\vee)$ . Moreover, if  $\mathcal{G} = \mathcal{O}_X(m)^{\oplus u}$  for some  $m$  and  $u$ , we can decompose  $\varphi$  into its components  $(\varphi_1, \dots, \varphi_u)$ . The condition of symmetricity means that the  $\varphi_i$  commute, i.e., for all  $i, j = 1, \dots, u$ ,  $(\varphi_i \otimes \text{id}_{\mathcal{G}}) \circ \varphi_j - (\varphi_j \otimes \text{id}_{\mathcal{G}}) \circ \varphi_i = 0$ .

As Yokogawa explains in [13], p.495, we can associate to a symmetric twisted endomorphism  $\varphi$  its characteristic polynomial in  $H^0(S^*(\mathcal{G})[t])$ . This provides us, in particular, with an element in the vector space  $\mathbb{H}_{\mathcal{G}} := \bigoplus_{i=1}^r H^0(S^i(\mathcal{G}))$  which we call abusively the *characteristic polynomial* of  $\varphi$ , too. Note that  $\varphi$  is not nilpotent if and only if its characteristic polynomial in  $\mathbb{H}_{\mathcal{G}}$  is not zero.

A framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H})$  will be called *symmetric*, if  $\varphi$  is symmetric. A symmetric framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H})$  defines an element in  $\mathbb{C} \oplus \mathbb{H}_{\mathcal{G}}$ . Now, let  $\mathbb{C}^*$  act on  $H^0(S^i(\mathcal{G}))$  through multiplication by  $z^i$ . This yields a  $\mathbb{C}^*$ -action on  $\mathbb{C} \oplus \mathbb{H}_{\mathcal{G}}$ , and the quotient  $\widehat{\mathbb{P}}_{\mathcal{G}}$  is a weighted projective space. Thus,  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  defines an element  $\widehat{\chi}(\mathcal{E}, \varepsilon, \varphi, \psi) \in \widehat{\mathbb{P}}_{\mathcal{G}}$  which depends only on its equivalence class. It will be referred to as *the characteristic polynomial* of  $(\mathcal{E}, \varepsilon, \varphi, \psi)$ .

*Characteristic vectors for framed Hitchin pairs.* Let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be a framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  with  $\mathcal{G} = \mathcal{O}_X(m)^{\oplus u}$ . Let  $T_1, \dots, T_s$  be the generators of the ring  $\mathbb{C}[M_r^{\oplus u}]^{\text{SL}_r}$ , belonging to the words  $\omega_1, \dots, \omega_s$ . For a given  $l$ , consider the homomorphism  $\varphi_l := (\varphi \otimes \text{id}_{\mathcal{G}^{\otimes l-1}}) \circ \dots \circ \varphi: \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{G}^{\otimes l}$ . Now, a word  $\omega$  of length  $l$  singles out a component of  $\varphi_l$ , i.e., a homomorphism  $\varphi_\omega: \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{O}_X(lm)$ , and the trace of  $\varphi_\omega$  gives a section in  $H^0(\mathcal{O}_X(lm))$ . For  $i = 1, \dots, s$ , let  $l_s$  be the length of the word  $\omega_s$ , and set  $\mathbb{K}_{\mathcal{G}} := \bigoplus_{i=1}^s H^0(\mathcal{O}_X(l_i m))$ . So, we can associate to  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  in a natural way an element in  $\mathbb{K}_{\mathcal{G}}$ . As before, we let  $\mathbb{C}^*$  act on  $H^0(\mathcal{O}_X(l_i m))$  through multiplication by  $z^{l_i}$ ,  $i = 1, \dots, s$ , in order to obtain a  $\mathbb{C}^*$ -action on  $\mathbb{C} \oplus \mathbb{K}_{\mathcal{G}}$ . The quotient  $(\mathbb{C} \oplus \mathbb{K}_{\mathcal{G}}) // \mathbb{C}^*$  will be denoted by  $\widehat{\Xi}_{\mathcal{G}}$ . Thus, we can assign to any framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  its *characteristic vector*  $\widehat{\xi}(\mathcal{E}, \varepsilon, \varphi, \psi) \in \widehat{\Xi}_{\mathcal{G}}$ . The characteristic vector clearly depends only on the equivalence class of  $(\mathcal{E}, \varepsilon, \varphi, \psi)$ .

**SEMISTABILITY AND SECTIONAL SEMISTABILITY.** Fix a polynomial  $\sigma \in \mathbb{Q}[t]$  with positive leading coefficient of degree at most  $\dim X - 1$  and call a framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$   $\sigma$ -(semi)stable, if for any proper, non-zero subsheaf  $\mathcal{F}$  of  $\mathcal{E}$  which is invariant under  $\varphi$ , i.e.,  $\varphi(\mathcal{F}) \subset \mathcal{F} \otimes \mathcal{G}$ ,

$$\begin{aligned} \frac{P_{\mathcal{F}}}{\text{rk } \mathcal{F}} - \frac{\sigma}{\text{rk } \mathcal{F}} & (\leq) \quad \frac{P_{\mathcal{E}}}{\text{rk } \mathcal{E}} - \frac{\sigma}{\text{rk } \mathcal{E}}, \quad \text{and} \\ \frac{P_{\mathcal{F}}}{\text{rk } \mathcal{F}} & (\leq) \quad \frac{P_{\mathcal{E}}}{\text{rk } \mathcal{E}} - \frac{\sigma}{\text{rk } \mathcal{E}}, \quad \text{if, furthermore, } \mathcal{F} \subset \ker \psi. \end{aligned}$$

*Remark 1.4.* i) Given two locally free sheaves  $\mathcal{G} \subset \mathcal{G}'$  and a framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H})$ , we can form the framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi', \psi)$  of type  $(P, \mathcal{G}', \mathcal{H})$  by defining  $\varphi'$  as the composition of  $\varphi$  with the inclusion  $\mathcal{E} \otimes \mathcal{G} \subset \mathcal{E} \otimes \mathcal{G}'$ . Since for any  $\varphi'$ -invariant subsheaf  $\mathcal{F}$  of  $\mathcal{E}$ , the map  $\mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{G}' \longrightarrow \mathcal{F} \otimes (\mathcal{G}'/\mathcal{G})$  is zero, it will also be  $\varphi$ -invariant. Therefore,  $(\mathcal{E}, \varepsilon, \varphi', \psi)$  will be  $\sigma$ -(semi)stable if and only if  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is  $\sigma$ -(semi)stable. In particular, since we can embed  $\mathcal{G}$  in  $\mathcal{O}_X(m)^{\oplus u}$  for some large  $m$  and  $u$ , we can and will always assume that  $\mathcal{G}$  is of that simple form.

ii) A  $\sigma$ -stable framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  has no automorphisms  $\rho$  besides the identity which satisfy

$$(\mathcal{E}, \varepsilon, \varphi, \psi) = (\mathcal{E}, \varepsilon, (\rho \otimes \text{id}_{\mathcal{G}}) \circ \varphi \circ \rho^{-1}, \psi \circ \rho^{-1}).$$

Next, fix a positive rational number  $\overline{\sigma}$ . A framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H})$  will be called  $\overline{\sigma}$ -sectional (semi)stable, if there is a subspace  $V \subset H^0(\mathcal{E})$  of dimension  $\chi(\mathcal{E})$ , s. th. for any non-trivial  $\varphi$ -invariant proper subsheaf  $\mathcal{F}$  of  $\mathcal{E}$

$$\begin{aligned} \frac{\dim(V \cap H^0(\mathcal{F}))}{\text{rk } \mathcal{F}} - \frac{\overline{\sigma}}{\text{rk } \mathcal{F}} & (\leq) \quad \frac{\chi(\mathcal{E})}{\text{rk } \mathcal{E}} - \frac{\overline{\sigma}}{\text{rk } \mathcal{E}}, \\ \frac{\dim(V \cap H^0(\mathcal{F}))}{\text{rk } \mathcal{F}} & (\leq) \quad \frac{\chi(\mathcal{E})}{\text{rk } \mathcal{E}} - \frac{\overline{\sigma}}{\text{rk } \mathcal{E}}, \quad \text{if } \mathcal{F} \subset \ker \psi. \end{aligned}$$

The usual arguments — assuming boundedness — then show:

**PROPOSITION 1.5.** *There is a natural number  $n_0$  such that for all  $n \geq n_0$  and all framed Hitchin pairs  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H})$  the following conditions are equivalent:*

1.  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is  $\sigma$ -(semi)stable.
2.  $(\mathcal{E}(n), \varepsilon, \varphi \otimes \text{id}_{\mathcal{O}_X(n)}, \psi \otimes \text{id}_{\mathcal{O}_X(n)})$  is  $\sigma(n)$ -sectional (semi)stable.
3.  $(\mathcal{E}(n), \varepsilon, \varphi \otimes \text{id}_{\mathcal{O}_X(n)}, \psi \otimes \text{id}_{\mathcal{O}_X(n)})$  satisfies the condition of  $\sigma(n)$ -sectional (semi) stability for globally generated subsheaves.

*Remark 1.6.* If  $X$  is curve and some positive rational number  $\sigma_\infty$ , then — as will follow from the results of the section about boundedness in the second chapter — one can choose  $n_0$  such that the above proposition holds true for all positive rational numbers  $\sigma < \sigma_\infty$ .

**JORDAN-HÖLDER FILTRATIONS AND S-EQUIVALENCE.** Let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be a  $\sigma$ -semi-stable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  which is not  $\sigma$ -stable. Let  $\mathcal{E}_1$  be a proper,  $\varphi$ -invariant, destabilizing subsheaf which is maximal w.r.t. inclusion. The homomorphisms  $\varphi$  and  $\psi$  can be restricted to  $\mathcal{E}_1$ , and  $(\mathcal{E}_1, \varepsilon, \varphi|_{\mathcal{E}_1}, \psi|_{\mathcal{E}_1})$  will be again  $\sigma$ -semistable. If it is not  $\sigma$ -stable, pick a maximal destabilizing  $\varphi$ -invariant subsheaf  $\mathcal{E}_2$  and so on in order to get a so called *Jordan-Hölder filtration*

$$0 =: \mathcal{E}_{m+1} \subset \mathcal{E}_m \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 := \mathcal{E}.$$

Then, we can define *the associated graded object*

$$\mathrm{gr}(\mathcal{E}, \varepsilon, \varphi, \psi) := \bigoplus_{i=1}^{m+1} (\mathcal{E}_{i-1}/\mathcal{E}_i, \varepsilon, \overline{\varphi}_i, \overline{\psi}_i).$$

As usual, this is well-defined up to equivalence. Two  $\sigma$ -semistable framed Hitchin pairs are called *S-equivalent* if their associated graded objects are equivalent, and a framed Hitchin pair is said to be  $\sigma$ -polystable, if it is equivalent to its associated graded object.

**THE MAIN RESULT.** Let  $\mathrm{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-(s)s}$  ( $\mathrm{FH}_{P/\mathcal{G}/\mathcal{H}/\mathrm{symm}}^{\sigma-(s)s}$ ) be the functors assigning to each noetherian scheme the set of equivalence classes of families of  $\sigma$ -(semi)stable (symmetric) framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H})$ .

**THEOREM 1.7.** i) *There are a quasi-projective scheme  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$  and a natural transformation  $\vartheta$  of  $\mathrm{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$  into the functor of points of  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$ , s. th. for any scheme  $\mathcal{M}$  and any natural transformation  $\vartheta': \mathrm{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss} \rightarrow h_{\mathcal{M}}$ , there is a unique morphism  $\tau: \mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss} \rightarrow \mathcal{M}$  with  $\vartheta' = h(\tau) \circ \vartheta$ .*

ii) *The space  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$  contains an open subscheme  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-s}$  which is a fine moduli space for the functor  $\mathrm{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-s}$ .*

iii) *The closed points of  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$  naturally correspond to the set of S-equivalence classes of  $\sigma$ -semistable framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H})$ .*

iv) *There is a proper morphism  $\hat{\xi}: \mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss} \rightarrow \hat{\Xi}_{\mathcal{G}}$  — called the generalized Hitchin map — which maps a  $\sigma$ -polystable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  to its characteristic vector. In particular,  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$  is a projective scheme.*

v) *There is a closed subscheme  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}/\mathrm{symm}}^{\sigma-ss}$  of  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$ , such that the schemes  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}/\mathrm{symm}}^{\sigma-ss}$  and  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}/\mathrm{symm}}^{\sigma-s} := \mathcal{FH}_{P/\mathcal{G}/\mathcal{H}/\mathrm{symm}}^{\sigma-ss} \cap \mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-s}$  enjoy the analogous properties to i) - iii) w.r.t. functors  $\mathrm{FH}_{P/\mathcal{G}/\mathcal{H}/\mathrm{symm}}^{\sigma-(s)s}$ . Moreover, there is a proper morphism  $\hat{\chi}: \mathcal{FH}_{P/\mathcal{G}/\mathcal{H}/\mathrm{symm}}^{\sigma-ss} \rightarrow \hat{\mathbb{P}}_{\mathcal{G}}$ , the Hitchin map, which maps a  $\sigma$ -polystable symmetric framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  to its characteristic polynomial.*

**PROOF OF THEOREM 1.7 WITH GIT.** We will follow the usual pattern of a GIT construction.

**Boundedness.** For a torsion free coherent sheaf  $\mathcal{E}$ , let  $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$  be its (slope) Harder-Narasimhan filtration. Set  $\mu_{\max}(\mathcal{E}) := \mu(\mathcal{E}_1)$  and  $\mu_{\min}(\mathcal{E}) := \mu(\mathcal{E}/\mathcal{E}_{l-1})$ . The proof of Nitsure [7], Proposition 3.2, can be easily extended to give the following

**PROPOSITION 1.8.** *Let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be a framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  with  $\mathcal{G} = \mathcal{O}_X(m)^{\oplus u}$ , such that there is a constant  $C \geq 0$  such that  $\mu(\mathcal{F}) \leq \mu(\mathcal{E}) + [C(r - \mathrm{rk} \mathcal{F})/(\mathrm{rk} \mathcal{F})]$  for any non-trivial  $\varphi$ -invariant subsheaf  $\mathcal{F}$  of  $\mathcal{E}$ . Then*

$$\mu_{\max}(\mathcal{E}) \leq \max \left\{ \mu + C, \mu + \frac{(r-1)^2}{r} \deg \mathcal{O}_X(m) \right\}.$$

*Remark 1.9.* As explained in [9], p.111, one can formulate a “polynomial” analogon to Proposition 1.8, namely, there is a constant  $C'$  such that for any subsheaf  $\mathcal{F}$  of  $\mathcal{E}$  one has  $P(\mathcal{F})/\mathrm{rk} \mathcal{F} < P(\mathcal{E})/\mathrm{rk} \mathcal{E} + C'x^{\dim X - 1}$ .

Since every  $\sigma$ -semistable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  satisfies the assumption of the above proposition with  $C =$  leading coefficient of  $\sigma$ , Maruyama’s boundedness result [6] yields

**COROLLARY 1.10.** *The isomorphism classes of torsion free coherent sheaves occuring in  $\sigma$ -semistable framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H})$  form a bounded family.*

Let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be a framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  and set  $\mathcal{F}_i := \ker((\varphi \otimes \mathrm{id}_{\mathcal{G}^{\otimes i-1}}) \circ \dots \circ \varphi)$ ,  $i = 1, \dots, r$ .

**LEMMA 1.11.** *The set of isomorphism classes of  $\mathcal{F}_i$ ’s coming from  $\sigma$ -semistable framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H})$  is also bounded.*

*Proof.* The proof of this lemma will be given below.  $\square$

*Some assumptions.* As observed in Remark 1.4, we can assume that  $\mathcal{G}$  is of the form  $\mathcal{O}_X(m)^{\oplus u}$  where  $\mathcal{O}_X(m)$  is globally generated, and, by Corollary 1.10 and Lemma 1.11, the following can be required.

**Assumptions 1.12.** Let  $n_1$  be a natural number such that for all  $n \geq n_1$  and every  $\sigma$ -semistable framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H})$  the following holds

- $\mathcal{H}(n)$  is globally generated and without higher cohomology.
- $\mathcal{E}(n)$  is globally generated and without higher cohomology.
- For  $i = 1, \dots, r$ , the sheaf  $\mathcal{F}_i(n)$  is globally generated and without higher cohomology.

An additional hypothesis on  $n_1$  will be explained later. Now, we have to make a ”logical loop”. Indeed, the last assumption makes use of 1.11 which we have not yet proved. Therefore, we won’t use this assumption in the following construction, prove Lemma 1.11, and then re-enter at the beginning of this section. Suppose also that  $n_1$  is greater than the constant  $n_0$  in Proposition 1.5. We may clearly assume that  $n_1 = 0$ .

*The parameter space.* Let  $V$  be a complex vector space of dimension  $p := P(0)$ . By Assumption 1.12, every torsion free sheaf  $\mathcal{E}$  occuring in a  $\sigma$ -semistable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  can be written as a quotient  $q: V \otimes \mathcal{O}_X \longrightarrow \mathcal{E}$  where  $H^0(q)$  is an isomorphism. These quotients are parametrized by a quasi-projective scheme  $\Omega_0$  which is an open subscheme of  $\Omega$ , the projective quot scheme of quotients of  $V \otimes \mathcal{O}_X$  which have Hilbert polynomial  $P$ . Let  $q_\Omega: V \otimes \mathcal{O}_{\Omega \times X} \longrightarrow \mathcal{E}_\Omega$  be the universal quotient on  $\Omega \times X$ . We can choose a  $v_0$  meeting the following requirements:

**Assumptions 1.13.** For any  $v \geq v_0$ , any subspace  $U$  of  $V$ , any  $[q: V \otimes \mathcal{O}_X \longrightarrow \mathcal{E}] \in \Omega$ , and  $\mathcal{E}_U := q(U \otimes \mathcal{O}_X)$ :

- $\mathcal{O}_X(v)$  is globally generated and without higher cohomology.
- The map  $H^0(\mathcal{O}_X(v)) \otimes H^0(\mathcal{O}_X(m)) \longrightarrow H^0(\mathcal{O}_X(v+m))$  is surjective.
- $H^i(\mathcal{E}_U(v)) = 0$ ,  $i > 0$ , and  $U \otimes \mathcal{O}_X(v) \longrightarrow H^0(\mathcal{E}_U(v))$  is surjective.

Hence,  $\pi_{\Omega*}(\mathcal{E} \otimes \pi_X^* \mathcal{O}_X(v_0))$  and  $\pi_{\Omega*}(\mathcal{E} \otimes \pi_X^* \mathcal{O}_X(v_0 + m))$  are locally free. Define

$$\widehat{\mathfrak{P}} := \mathbb{P}(\mathcal{O}_\Omega \oplus \pi_{\Omega*}(\mathcal{E} \otimes \pi_X^* \mathcal{O}_X(v_0 + m))^{\oplus u} \otimes \pi_{\Omega*}(\mathcal{E} \otimes \pi_X^* \mathcal{O}_X(v_0))).$$

There is a tautological line bundle  $\mathfrak{N}_{\widehat{\mathfrak{P}}}$  on  $\widehat{\mathfrak{P}}$ , and the tautological surjection provides us, on  $\widehat{\mathfrak{P}} \times X$ , with a homomorphism

$$V \otimes \pi_X^* \mathcal{O}_X(v_0) \longrightarrow (\mathcal{E}_{\widehat{\mathfrak{P}}} \otimes \pi_{\widehat{\mathfrak{P}}}^* \mathfrak{N}_{\widehat{\mathfrak{P}}} \otimes \pi_X^* \mathcal{O}_X(v_0 + m))^{\oplus u} \otimes \pi_{\widehat{\mathfrak{P}}}^* \mathfrak{N}_{\widehat{\mathfrak{P}}}.$$

Here,  $q_{\widehat{\mathfrak{P}}} : V \otimes \mathcal{O}_{\widehat{\mathfrak{P}} \times X} \longrightarrow \mathcal{E}_{\widehat{\mathfrak{P}}}$  is the pullback of the universal quotient. Define  $\mathfrak{P}$  as the closed subscheme of  $\widehat{\mathfrak{P}}$  where this homomorphism factorizes through  $\mathcal{E}_{\widehat{\mathfrak{P}}} \otimes \pi_X^* \mathcal{O}_X(v_0)$ . Let

$$\varphi_{\mathfrak{P}} : \mathcal{E}_{\mathfrak{P}} \longrightarrow (\mathcal{E}_{\mathfrak{P}} \otimes \pi_X^* \mathcal{O}_X(m))^{\oplus u} \otimes \pi_{\mathfrak{P}}^* \mathfrak{N}_{\mathfrak{P}}$$

be the induced homomorphism and  $\mathfrak{N}_{\mathfrak{P}}$  the restriction of  $\mathfrak{N}_{\widehat{\mathfrak{P}}}$  to  $\mathfrak{P}$ . By Assumption 1.12, for any  $\sigma$ -(semi)stable framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$ , the homomorphism  $\psi : \mathcal{E} \longrightarrow \mathcal{H}$  is determined by the homomorphism  $H^0(\varphi) : H^0(\mathcal{E}) \longrightarrow H^0(\mathcal{H})$ . Define the space  $\mathbf{R} := \mathbb{P}(\text{Hom}(V, H^0(\mathcal{H})))^\vee$ . It is now clear how to construct the parameter space  $\mathfrak{R}$  as a closed subscheme of  $\mathfrak{P} \times \mathbf{R}$ , and that, on  $\mathfrak{R} \times X$ , there is a universal family  $(\mathcal{E}_{\mathfrak{R}}, \varepsilon_{\mathfrak{R}}, \varphi_{\mathfrak{R}}, \psi_{\mathfrak{R}}, \mathfrak{N}_{\mathfrak{R}})$ , such that any family of  $\sigma$ -semistable framed Hitchin pairs can locally be obtained as the pullback of this universal family.  $\mathfrak{R}_0$  will be the open subscheme sitting over  $\Omega_0$ , and  $\mathfrak{R}_0^{\sigma-(s)}$  will be the — a posteriori — open subscheme parametrizing  $\sigma$ -(semi)stable framed Hitchin pairs.

*Ample line bundles on  $\widehat{\mathfrak{P}}$ .* For the moment, write  $\mathfrak{W}_{\Omega}$  for the vector bundle  $\pi_{\Omega*}(\mathcal{E}_{\Omega} \otimes \pi_X^* \mathcal{O}_X(v_0 + m))$ . Note  $\mathfrak{W}_{\Omega}^\vee \cong \bigwedge^{P(v_0+m)-1} \mathfrak{W}_{\Omega} \otimes \det \mathfrak{W}_{\Omega}^\vee$ . In particular, there is a surjection

$$\bigwedge^{P(v_0+m)-1} (V \otimes N \otimes M) \otimes \det \mathfrak{W}_{\Omega}^\vee \longrightarrow \mathfrak{W}_{\Omega}^\vee.$$

The line bundle  $\mathcal{L}_{\Omega} := \det \mathfrak{W}_{\Omega}$  on  $\Omega$  is very ample. Therefore, we see that  $\mathcal{O}_{\widehat{\mathfrak{P}}}(a_1, a_2) := \pi^* \mathcal{L}_{\Omega}^{\otimes a_1} \otimes \mathfrak{N}_{\widehat{\mathfrak{P}}}^{\otimes a_2}$  is globally generated for  $a_1 \geq a_2 > 0$  and very ample for  $a_1 > a_2 > 0$ .

*The group actions.* There are natural right actions of  $\text{SL}(V)$  on the schemes  $\Omega$  and  $\mathbf{R}$ , and the locally free sheaves  $\pi_{\Omega*}(\mathcal{E}_{\Omega} \otimes \pi_X^* \mathcal{O}_X(v_0))$  and  $\pi_{\Omega*}(\mathcal{E}_{\Omega} \otimes \pi_X^* \mathcal{O}_X(v_0 + m))$  are naturally linearized w.r.t. the group action on  $\Omega$ . Thus, there is a natural  $\text{SL}(V)$ -action from the right on  $\widehat{\mathfrak{P}} \times \mathbf{R}$ . This leaves the schemes  $\mathfrak{R}$  and  $\mathfrak{R}_0$  invariant. Moreover, the equivalence relation on the closed points of  $\mathfrak{R}_0$  induced by this group action is just the relation  $\sim$  on framed Hitchin pairs. More precisely,

**PROPOSITION 1.14.** *For any noetherian scheme  $S$  and any two morphisms  $\beta_i : S \longrightarrow \mathfrak{R}_0$ ,  $i = 1, 2$ , such that the pullbacks of the universal family via  $(\beta_1 \times \text{id}_X)$  and  $(\beta_2 \times \text{id}_X)$  are equivalent, there exist an étale covering  $\tau : T \longrightarrow S$  and a morphism  $\Gamma : T \longrightarrow \text{SL}(V)$  with  $\beta_1 \circ \tau = (\beta_2 \circ \tau) \cdot \Gamma$ .*

*Remark 1.15.* Using the universal properties, it is easy to see that the universal family on  $\mathfrak{R} \times X$  can be equipped with an  $\text{SL}(V)$ -linearization. Now, restrict everything to  $\mathfrak{R}_0^{\sigma-s}$ . Then, by Remark 1.4 ii), the  $\text{SL}(V)$ -linearization induces a  $\text{PGL}(V)$ -linearization of the universal family. Since all the  $\text{PGL}(V)$ -stabilizers are trivial, the universal family descends to the quotient  $\mathfrak{R}_0^{\sigma-s} // \text{SL}(V) = \mathfrak{R}_0^{\sigma-s} // \text{PGL}(V)$  provided the latter space exists. For the necessary descent theory, the reader is referred to [2], p.87.

*The semistable points in the closure of  $\mathfrak{R}_0$  and the proof of the main theorem.* Fix the polarization  $\mathcal{O}(2, 1, a)$  (compare [1], p.309) on  $\mathfrak{R}$  with

$$\frac{a}{2} := (P(v_0 + m) - \sigma(v_0 + m)) \frac{\sigma(0)}{p - \sigma(0)} - \sigma(v_0 + m).$$

The group action will be linearized in that line bundle.

**THEOREM 1.16.** *Let  $r := ([q : V \otimes \mathcal{O}_X \longrightarrow \mathcal{E}], [\varepsilon, \varphi], [\psi])$  be a point in the parameter space  $\mathfrak{R}_0$ . Then,  $r$  is (semi/poly)stable in  $\mathfrak{R}$  (w.r.t. the chosen linearization) if and only if  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is a  $\sigma$ -(semi/poly)stable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$ . Moreover, if  $X$  is a curve, and  $r$  is a point in the closure of the parameter space  $\mathfrak{R}_0$ , then  $r$  is (semi/poly)stable in  $\mathfrak{R}$  if and only if  $r$  lies in  $\mathfrak{R}_0$  and  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is a  $\sigma$ -(semi/poly)stable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$ .*



*Proof.* This theorem will be proved below.  $\square$

If  $X$  is a curve, then Theorem 1.16 shows that  $\mathfrak{R}_0^{\sigma-(s)s}$  is exactly the set of (semi)stable points in the closure of  $\mathfrak{R}_0$ . Thus, the good quotients  $\mathfrak{R}_0^{\sigma-(s)s} // \mathrm{SL}(V)$  do exist, the space  $\mathfrak{R}_0^{\sigma-ss} // \mathrm{SL}(V)$  is a projective scheme, and  $\mathfrak{R}_0^{\sigma-s} // \mathrm{SL}(V)$  is a geometric quotient. In case  $X$  is higher dimensional, Theorem 1.16 shows that  $\mathfrak{R}_0^{\sigma-(s)s}$  is a saturated open subet of the semistable points in  $\overline{\mathfrak{R}_0}$ , i.e., the closure in  $\overline{\mathfrak{R}_0^{ss}}$  of the orbit of a point in  $\mathfrak{R}_0^{ss}$  still lies in  $\mathfrak{R}_0^{ss}$ . Therefore, the good (geometric) quotient  $\mathfrak{R}_0^{\sigma-(s)s} // \mathrm{SL}(V)$  exists as a quasi-projective scheme. Defining  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-(s)s} := \mathfrak{R}_0^{\sigma-(s)s} // \mathrm{SL}(V)$  gives our moduli spaces. The first assertion of Theorem 1.7 is then — as usual — a direct consequence of the local universal property of  $\mathfrak{R}$ , Proposition 1.14, and the universal property of the categorical quotient. As we have already remarked in 1.15, the universal family on  $\mathfrak{R}_0^{\sigma-s} \times X$  descends to  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-s} \times X$ , whence  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-s}$  is a fine moduli space. The identification of the closed points follows from the assertion about the polystable points in Theorem 1.16. Thus, i) - iii) in 1.7 are settled. For point v), we define  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}/\mathrm{symm}}^{\sigma-ss}$  as the image of  $\mathfrak{R}_{0,\mathrm{symm}}^{\sigma-ss}$  in  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$ . This space clearly has the desired properties and coincides with  $\mathfrak{R}_{0,\mathrm{symm}}^{\sigma-ss} // \mathrm{SL}(V)$ . To define the generalized Hitchin map, let  $\hat{\mathfrak{H}}$  be the geometric vector bundle associated to the locally free sheaf

$$\pi_{\Omega*}(\mathfrak{E} \otimes \pi_X^* \mathcal{O}_X(v_0))^\vee \otimes \pi_{\Omega*}(\mathfrak{E} \otimes \pi_X^* \mathcal{O}_X(v_0 + m))^{\oplus u}.$$

The induced map  $([\mathbb{C} \times \hat{\mathfrak{H}}] \setminus \{\text{zero section}\}) \times \mathbf{R} \longrightarrow \hat{\mathfrak{P}} \times \mathbf{R}$  is a good  $\mathbb{C}^*$ -quotient. Let  $\mathfrak{R}_0^{\sigma-ss}$  be the preimage of  $\mathfrak{R}_0^{\sigma-ss}$  under the above map. Then,  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss} = \mathfrak{R}_0^{\sigma-ss} // (\mathbb{C}^* \times \mathrm{SL}(V))$  is also a good quotient. Copying the construction of [13], p.496, we obtain a morphism  $\mathfrak{R}_0^{\sigma-ss} \longrightarrow \mathbb{C} \times \mathbb{K}_{\mathcal{G}} \longrightarrow \hat{\Xi}_{\mathcal{G}}$  which is  $(\mathbb{C}^* \times \mathrm{SL}(V))$ -invariant and, thus, descends to a morphism

$$\hat{\xi}: \mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss} \longrightarrow \hat{\Xi}_{\mathcal{G}}.$$

This is the generalized Hitchin map. To see that it is proper, let  $(C, 0)$  be the spectrum of a discrete valuation ring  $R$  with field of fractions  $K$ . Suppose there is a map  $C \longrightarrow \hat{\Xi}_{\mathcal{G}}$  which lifts via  $\hat{\xi}$  over  $C \setminus 0$  to  $\mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$ . Since there are no non-trivial line bundles on  $C$ , the morphism from  $C$  to  $\hat{\Xi}_{\mathcal{G}}$  lifts to  $\mathbb{C} \oplus \mathbb{K}_{\mathcal{G}}$ . After possibly passing to a finite extension of  $K$ , we may assume that the map from  $C \setminus 0 \longrightarrow \mathcal{FH}_{P/\mathcal{G}/\mathcal{H}}^{\sigma-ss}$  comes from a family of  $\sigma$ -semistable framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H})$  and that the induced map to  $\mathbb{K}_{\mathcal{G}}$  is just the characteristic vector of that family. This follows from Luna's étale slice theorem and our definition of equivalence of families. It is not hard to see that the arguments used by Yokogawa ([13], p.487ff) or Nitsure [7] can be adapted to our situation. We omit this here, because it does not involve any new idea.  $\square$

*Proof of Theorem 1.16.* First, we prove the statement about the (semi)stable points, using the Hilbert-Mumford criterion. For this, we recall that a one parameter subgroup of  $\mathrm{SL}(V)$  is determined by giving a basis  $v_1, \dots, v_p$  of  $V$  and weights  $\gamma_1 \leq \dots \leq \gamma_p$ , satisfying  $\sum_i \gamma_i = 0$ . Moreover, a weight vector  $(\gamma_1, \dots, \gamma_p)$  with  $\gamma_1 \leq \dots \leq \gamma_p$  and  $\sum_i \gamma_i = 0$  can be written as

$$(\gamma_1, \dots, \gamma_p) = \sum_{i=1}^{p-1} \frac{\gamma_{i+1} - \gamma_i}{p} \gamma^{(i)}$$

with  $\gamma^{(i)} = (i-p, \dots, i-p, i, \dots, i)$  where  $i-p$  occurs  $i$  times. Let us first make some preliminary remarks.

*The weights of the  $\mathbf{R}$ -component.* Let  $\lambda$  be the one parameter subgroup which is determined by the basis  $v_1, \dots, v_p$  and the weight vector  $(\gamma_1, \dots, \gamma_p)$ . Let  $[h] \in \mathbf{R}$  be a point. It follows that  $\mu_{\mathbf{R}}([h], \lambda) = -\min\{\gamma_i \mid h(v_i) \neq 0\}$ . In particular, if  $\lambda^{(i)}$  is the one parameter subgroup defined by  $\gamma^{(i)}$ , then  $\mu_{\mathbf{R}}([h], \lambda^{(i)}) = -i$  or  $p-i$ , depending on whether  $\langle v_1, \dots, v_i \rangle$  is contained in  $\ker(h)$  or not. Observe that  $\mu_{\mathbf{R}}([h], \lambda_1 \cdot \lambda_2) = \mu_{\mathbf{R}}([h], \lambda_1) + \mu_{\mathbf{R}}([h], \lambda_2)$  for any two one parameter subgroups given w.r.t. above basis by the weights  $\gamma'_1 \leq \dots \leq \gamma'_p$ ,  $i = 1, 2$ .

*The weights of  $\mathfrak{L}_{\Omega}$ .* Let  $v_1, \dots, v_p$  be a basis for  $V$  and  $[q] \in \Omega$ . Define  $Q_i := H^0(q \otimes \text{id}_{\mathcal{O}_X(v_0+m)})(\langle v_1, \dots, v_i \rangle \otimes N \otimes H^0(\mathcal{O}_X(m)))$ , and  $\delta(i) := \dim Q_i$ ,  $i = 1, \dots, p$ . For a one parameter subgroup given w.r.t. the above basis by the weights  $\gamma_1 \leq \dots \leq \gamma_p$ , we obtain

$$\mu_{\Omega}([q], \lambda) = -\sum_{i=1}^p (\delta(i) - \delta(i-1))\gamma_i,$$

in particular,  $\mu_{\Omega}([q], \lambda^{(i)}) = (p\delta(i) - iP(v_0+m))$ . Again,  $\mu_{\Omega}([q], \lambda_1 \cdot \lambda_2) = \mu_{\Omega}([q], \lambda_1) + \mu_{\Omega}([q], \lambda_2)$ .

*The global sections of  $\mathcal{O}_{\mathfrak{P}}(1, 1)$ .* All the assertions about the weights of one parameter subgroups will rely on a good understanding of the global sections of the line bundle  $\mathcal{O}_{\mathfrak{P}}(1, 1)$  over  $\mathfrak{P}$ . Therefore, we will now give an explicit description of them. Let  $[q: V \otimes \mathcal{O}_X \rightarrow \mathcal{E}]$  be a point in  $\Omega$ . The points of  $\widehat{\mathfrak{P}}$  in the fibre over  $[q]$  can then be written as classes  $[\varepsilon, f]$ ,  $\varepsilon \in \mathbb{C}$ ,  $f \in H := \text{Hom}(H^0(\mathcal{E}(v_0)), H^0(\mathcal{E}(v_0+m))^{\oplus u})$ , and  $[\varepsilon, f] = [z\varepsilon, zf]$ ,  $z \in \mathbb{C}^*$ . Now, fix bases  $v_1, \dots, v_p$  of  $V$ ,  $n_1, \dots, n_v$  of  $N$ , and  $m_1, \dots, m_{\mu}$  of  $H^0(\mathcal{O}_X(m))$ . Using the lexicographic order, we get ordered bases for  $V \otimes N$  and  $V \otimes N \otimes H^0(\mathcal{O}_X(m))$ . Set  $p' := P(v_0)$  and  $p'' := P(v_0+m)$ . Let  $\mathfrak{I}$  be the set of all  $p'$ -tuples of elements of the form  $v_i \otimes n_{\kappa}$  whose images in  $H^0(\mathcal{E}(v_0))$  form a basis. Likewise  $\mathfrak{J}$  is defined as the set of  $p''$ -tuples of the form  $v_i \otimes n_{\kappa} \otimes m_{\lambda}$  which induce a basis for  $H^0(\mathcal{E}(v_0+m))$ . Observe that specifying an element in  $J \in \mathfrak{J}$  is the same as specifying an element  $S_J$  in  $\bigwedge^{p''} V \otimes N \otimes H^0(\mathcal{O}_X(m))$  which, viewed as a global section of  $\mathfrak{L}_{\Omega}$ , does not vanish in  $[q]$ . Pick elements  $I \in \mathfrak{I}$  and  $J \in \mathfrak{J}$ , and let  $u_1, \dots, u_{p'}$  and  $w_1, \dots, w_{p''}$  be the induced bases of  $H^0(\mathcal{E}(v_0))$  and  $H^0(\mathcal{E}(v_0+m))$ , respectively. We, thus, obtain a basis  $w_1^1, \dots, w_{p''}^1, \dots, w_1^{p'}, \dots, w_{p''}^{p'}$  for  $H^0(\mathcal{E}(v_0+m))^{\oplus u}$ . The  $f_{ij}^k := u_i^{\vee} \otimes w_j^k$ ,  $i = 1, \dots, p'$ ,  $j = 1, \dots, p''$ ,  $k = 1, \dots, u$ , form a basis for  $H$ . As one knows from linear algebra,  $w_j^{k\vee}$  can be identified — up to a sign — with  $(w_1^k \wedge \dots \wedge w_j^k \wedge \dots \wedge w_{p'}^k) / (w_1^k \wedge \dots \wedge w_{p'}^k)$ . Thus,  $u_i \otimes w_j^{k\vee}$  defines a rational section  $\sigma_{IJ}^{ijk}$  of  $V \otimes N \otimes \bigwedge^{p''-1}(V \otimes N \otimes M) \otimes \mathfrak{L}_{\Omega}^{\vee}$ . Therefore,  $\tilde{\Theta}_{IJ}^{ijk} := \sigma_{IJ}^{ijk} \otimes S_J$  is a global section of  $V \otimes N \otimes \bigwedge^{p''-1}(V \otimes N \otimes M) \otimes \mathcal{O}_{\widehat{\mathfrak{P}}}$ . Denote the induced section of  $\mathcal{O}_{\widehat{\mathfrak{P}}}(1, 1)$  by  $\Theta_{IJ}^{ijk}$ . From the construction, it is clear that the  $\Theta_{IJ}^{ijk}$  generate  $\mathcal{O}_{\widehat{\mathfrak{P}}}(1, 1)$  everywhere and that they are eigenvectors for the action of the maximal torus defined by the basis  $v_1, \dots, v_p$ . Let's return to the original setting. Let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be a framed Hitchin pair. Call a subsheaf  $\mathcal{F}$  of  $\mathcal{E}$   $\varphi$ -superinvariant, if  $\mathcal{F} \subset \ker \psi$  and the induced homomorphism  $\mathcal{E}/\mathcal{F} \rightarrow (\mathcal{E}/\mathcal{F}) \otimes \mathcal{G}$  is also zero. As an immediate consequence of the previous discussion, we note

LEMMA 1.17. *Fix a basis  $v_1, \dots, v_p$  for  $V$ . Then one obtains the following values for the action of  $\lambda^{(i)}$  on the fibre of  $\mathfrak{N}_{\mathfrak{P}}$  over  $\lim_{z \rightarrow 0} \mathbf{p} \cdot \lambda^{(i)}(z)$  with  $\mathbf{p} = ([q: V \otimes \mathcal{O}_X \rightarrow \mathcal{E}], [\varepsilon, \varphi])$ ,  $i = 1, \dots, p$ : i)  $-p$  if  $\mathcal{E}_{\langle v_1, \dots, v_i \rangle}$  is not  $\varphi$ -invariant; ii)  $p$  if  $\varepsilon = 0$  and  $\mathcal{E}_{\langle v_1, \dots, v_i \rangle}$  is  $\varphi$ -superinvariant; and iii)  $0$  in all the other cases.*

The above description of the weights and a standard argument in Simpson-type constructions — [1] being closest to our situation — lead to the following conclusion:

PROPOSITION 1.18. *Let  $r = ([q: V \otimes \mathcal{O}_X \rightarrow \mathcal{E}], [\varepsilon, \varphi], [\psi])$  be in the closure of  $\mathfrak{R}_0$ . Then the following assertions are equivalent:*

1. For any basis  $v_1, \dots, v_p$  of  $V$  and any one parameter subgroup  $\lambda^{(i)}$ , such that the sheaf  $\mathcal{E}_{\langle v_1, \dots, v_i \rangle}$  is  $\varphi$ -invariant but not  $\varphi$ -superinvariant when  $\varepsilon = 0$ ,  $\mu(r, \lambda^{(i)}) \geq 0$ .
2. For all subspaces  $U \subset V$  such that  $\mathcal{E}_U$  is  $\varphi$ -invariant but not  $\varphi$ -superinvariant when  $\varepsilon = 0$ ,

$$\begin{aligned} \frac{\dim U}{\operatorname{rk} \mathcal{E}_U} - \frac{\sigma(0)}{\operatorname{rk} \mathcal{E}_U} & (\leq) \quad \frac{p}{r} - \frac{\sigma(0)}{r} \\ \frac{\dim U}{\operatorname{rk} \mathcal{E}_U} & (\leq) \quad \frac{p}{r} - \frac{\sigma(0)}{r}, \quad \text{if } \mathcal{E}_U \subset \ker \varphi. \end{aligned}$$

Now, we proceed to the proof of Theorem 1.16. Let's start with the following obvious

**COROLLARY 1.19.** *Suppose  $X$  is a curve and  $r$  lies in  $\overline{\mathfrak{H}_0}$ . Then  $H^0(q)$  must be an isomorphism and  $\mathcal{E}$  must be torsion free.*

*Remark 1.20.* In the higher dimensional cases, one cannot adapt the proof for the moduli spaces of semistable sheaves. First, one can copy the proof of Proposition 4.4.2 in [2] to get a homomorphism  $\kappa: \mathcal{E} \rightarrow \mathcal{E}'$ . One can even equip  $\mathcal{E}'$  with the structure of a framed Hitchin pair such that  $\kappa$  becomes a homomorphism between framed Hitchin pairs. But unfortunately, it is not clear whether  $\ker(V \rightarrow H^0(\mathcal{E}'))$  generates a  $\varphi$ -invariant (torsion) subsheaf of  $\mathcal{E}$ .

We will apply the Criterion iii) of Proposition 1.5. First, assume that  $r$  is a (semi)stable point. Then, by Proposition 1.18, we only have to show that  $\varphi$  can't be nilpotent when  $\varepsilon = 0$ . If  $\varepsilon = 0$  and  $\varphi$  is nilpotent, then there is a filtration

$$0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_s \subset \mathcal{F}$$

such that  $\varphi(\mathcal{F}_i) \subset \mathcal{F}_{i-1} \otimes \mathcal{G}$  and the  $\mathcal{F}_i$  are globally generated,  $i = 1, \dots, s$  (Assumption 1.12). Choose a basis  $v_1, \dots, v_p$  such that there are indices  $t_1 < \dots < t_s$  with  $\langle v_1, \dots, v_{t_i} \rangle = H^0(\mathcal{F}_i)$ ,  $i = 1, \dots, s$ . Let  $\lambda$  be given w.r.t. that basis by  $\sum_{i=1}^s \gamma^{(t_i)}$ . Then, semistability and Assumption 1.12 yield

$$-p + 2 \sum_{i=1}^s \left( p h^0(\mathcal{F}_i(v_0 + m)) - h^0(\mathcal{F}_i) P(v_0 + m) + \frac{a}{2} (p - h^0(\mathcal{F}_s)) \right) \geq 0.$$

Plugging in our formula for  $a/2$ , viewing everything as polynomials in  $v_0$  and taking leading coefficients gives

$$-p + \sigma(0) + 2 \sum_{i=1}^s \left( p \operatorname{rk} \mathcal{F}_i - h^0(\mathcal{F}_i) r + \sigma(0)(r - \operatorname{rk} \mathcal{F}_i) \right) \geq 0.$$

If we have chosen  $n_1$  in 1.12 big enough, then this is not a possibility. Indeed,  $-P + \sigma$  — as a function of  $n$  — is a polynomial of degree  $\dim X$  with negative leading coefficient whereas the polynomial corresponding to the sum has at most degree  $\dim X - 1$ . Since, by Corollary 1.10, there can occur only finitely many such polynomials, we are done.

To see the converse, let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  satisfy Condition iii) of Prop. 1.5. Then, the second condition in Proposition 1.18 is satisfied. Let  $v_1, \dots, v_p$  be any basis and  $\lambda$  be a one parameter subgroup given by, say,  $\sum_{i=1}^{p-1} \alpha_i \gamma^{(i)}$ ,  $\alpha_i \in \mathbb{Z}[(1/p)]_{\geq 0}$ . Note that, if  $\varphi$  is not nilpotent, there is a non-zero global section of  $\mathcal{O}(2, 1, a)$  on which every one parameter subgroup  $\lambda$  acts with weight  $\leq -2\mu_{\Omega}([q], \lambda) - a\mu_{\mathbf{R}}([\psi], \lambda)$ . Therefore, if  $\mathcal{E}_{\langle v_1, \dots, v_i \rangle}$  is  $\varphi$ -invariant for every  $i$  with  $\alpha_i \neq 0$ , there is nothing to show. Otherwise let  $i_1, \dots, i_s$  be the indices belonging to non-invariant subsheaves. For each  $j = 1, \dots, s$ , let  $\mathcal{E}_{i_j}$  be the saturation of  $\mathcal{E}_{V_{i_j}}$ . We then find indices  $t_1, \dots, t_t$  among  $i_1, \dots, i_s$  such that  $\mathcal{E}_{V_{i_j}} \subset \mathcal{E}_{t_k}$  if and only if  $i_j \leq t_k$ ,  $j = 1, \dots, s$ ,  $k = 1, \dots, t$ . For  $t_{k-1} < i_j \leq t_k$ , the induced homomorphism  $\mathcal{E}_{V_{i_j}} \rightarrow \mathcal{E}/\mathcal{E}_{t_k} \otimes \mathcal{G}$  will be non-zero. Therefore, we find a section  $\Theta_k$  in  $H^0(\mathfrak{P}, \mathcal{O}_{\mathfrak{P}}(1, 1))$  among the  $\Theta_{IJ}^{ijk}$  such that  $\lambda^{(i_j)}$  acts on  $\Theta_k$  with weight  $-p - \mu_{\Omega}([q], \lambda^{(i_j)})$  for every  $j$

with  $\iota_{\kappa-1} < i_j \leq \iota_\kappa$  and any other one parameter group  $\lambda^{(i)}$  with weight  $\leq -\mu_\Omega([q], \lambda^{(i)})$ . Let  $\Theta := \Theta_1 \otimes \cdots \otimes \Theta_t$  be the corresponding section of  $\mathcal{O}_{\mathfrak{P}}(t, t)$ . Then  $\lambda$  acts on  $\Theta$  with weight  $\leq -p \sum_{j=1}^s \alpha_{i_j} - t \sum_{i=1}^p \alpha_i \mu_\Omega([q], \lambda^{(i)})$ . Therefore, the assertion  $\mu(r, \lambda)(\geq) 0$  can be reduced to

$$p + 2t(\mu_\Omega([q], \lambda^{(i_j)}) + \frac{a}{2}\mu_{\mathbf{R}}([\psi], \lambda^{(i_j)})) > 0, \quad j = 1, \dots, s.$$

Now, only those  $i_j$  matter for which  $\mu_\Omega([q], \lambda^{(i_j)}) + (a/2)\mu_{\mathbf{R}}([\psi], \lambda^{(i_j)})$  is negative. But then,  $\mathcal{E}_{V_{i_j}}$  can be assumed to be globally generated and without higher cohomology, and replacing this sheaf with its saturation, we are left to show

$$p - \sigma(0) + 2t(\text{rk } \mathcal{E}_{\iota_\tau}(p - \sigma(0)) - rh^0(\mathcal{E}_{\iota_\tau})) > 0, \quad \tau = 1, \dots, t.$$

We view this again as an inequality between polynomials in  $n$ . By Remark 1.9, the second term is then  $\geq -2t(r-1)[rC'n^{\dim X-1} + \sigma(n)]$ , i.e., bounded from below by a polynomial of degree  $\dim X - 1$ , and  $P(n) - \sigma(n)$  is a polynomial of degree  $\dim X$  with positive leading coefficient, whence the claim.

For the assertion about the polystable points, let the one parameter subgroup  $\lambda$  be given by  $\sum_{i=1}^{p-1} \alpha_i \gamma^{(i)}$ ,  $\alpha_i \in \mathbb{Z}[(1/p)]_{\geq 0}$ , w.r.t. the basis  $v_1, \dots, v_p$  of  $V$ . Observe that the above proof shows that  $\mu(\lambda, r) = 0$  can only occur if for each  $\alpha_i \neq 0$ ,  $\langle v_1, \dots, v_i \rangle =: V_i = H^0(\mathcal{E}_{V_i})$ ,  $\mathcal{E}_{V_i}$  is  $\varphi$ -invariant and destabilizes the framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$ . So,  $r$  will be a fixed point for the action of any such  $\lambda$  if and only if  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is a  $\sigma$ -polystable framed Hitchin pair.  $\square$

**SOME VARIATIONS AND EXAMPLES.** In this section, let  $X$  be a curve, and  $L = \mathcal{G}$  a line bundle. The type will be written as  $(d, r, L, \mathcal{H})$ .

*Framed Hitchin pairs with fixed determinant.* Fix a line bundle  $M$  of degree  $d$ . A *framed Hitchin pair of type  $(M, r, L, \mathcal{H})$*  is a framed Hitchin pair  $(E, \varepsilon, \varphi, \psi)$  of type  $(d, r, L, \mathcal{H})$  with  $\det(E) \cong M$ . The equivalence relation is the same as for framed Hitchin pairs of type  $(d, r, L, \mathcal{H})$ . Observe that the sheaf  $\mathfrak{E}_{\mathfrak{R}}$  on  $\mathfrak{R}$  provides us with a morphism  $d(\mathfrak{E}_{\mathfrak{R}}): \mathfrak{R} \rightarrow \text{Pic } X$ . Since this morphism is  $\text{SL}(V)$ -invariant, it descends to a map  $\mathfrak{d}: \mathcal{F}\mathcal{H}_{d/r/L/\mathcal{H}}^{\sigma-ss} \rightarrow \text{Pic } X$ . Define  $\mathcal{F}\mathcal{H}_{M/r/L/\mathcal{H}}^{\sigma-ss}$  as the scheme theoretic fibre of  $\mathfrak{d}$  over  $[M]$ .

*The condition  $\text{Im}(\varphi) \subset \ker(\psi) \otimes L$ .* Stupariu looks at framed Hitchin pairs which satisfy  $(\psi \otimes \text{id}_L) \circ \varphi = 0$ , i.e.,

$$\text{Im} \varphi \subset (\ker \psi) \otimes L.$$

Let us call these objects *framed Hitchin pairs of type  $(d, r, L, \mathcal{H}, *)$* . The definitions of *equivalence*,  $\sigma$ -(semi)stability, and so on carry over. Note that the above condition forces  $\ker \psi$  to be  $\varphi$ -invariant. We can also construct moduli spaces for those objects: Consider, on  $\mathfrak{R} \times X$ ,

$$\pi_X^* \mathcal{H}^{\vee} \xrightarrow{\psi_{\mathfrak{R}}^{\vee}} \mathfrak{E}_{\mathfrak{R}}^{\vee} \xrightarrow{\varphi_{\mathfrak{R}}^{\vee} \otimes \text{id}} \pi_{\mathfrak{R}}^* \mathfrak{N}_{\mathfrak{R}} \xrightarrow{\otimes \text{id}} \pi_X^* L \otimes \pi_{\mathfrak{R}}^* \mathfrak{N}_{\mathfrak{R}} \otimes \pi_X^* L.$$

Define  $\mathfrak{R}^*$  as the closed subscheme of  $\mathfrak{R}$  whose closed points are the points  $r \in \mathfrak{R}$  such that the above homomorphism becomes zero when restricted to  $\{r\} \times X$ . One can now go on as before. We denote the resulting moduli spaces by  $\mathcal{F}\mathcal{H}_{d/r/L/\mathcal{H}/*}^{\sigma-(s)s}$ . Moreover, we can fix a line bundle  $M$  and define the moduli spaces  $\mathcal{F}\mathcal{H}_{M/r/L/\mathcal{H}/*}^{\sigma-(s)s}$ .

*Some observations.* Let  $(E, \varepsilon, \varphi, \psi)$  be of type  $(d, 2, L, \mathcal{O}_X(m_0), *)$ ,  $m_0 \in \mathbb{Z}$ . The image of  $\psi$  is of the form  $\mathcal{O}_X(m_0)(-D)$  for some effective divisor  $D$ , and we get an extension

$$(e): 0 \rightarrow \det(E)(D)(-m_0) \rightarrow E \rightarrow \mathcal{O}_X(m_0)(-D) \rightarrow 0.$$

LEMMA 1.21. i) If  $\sigma > 2m_0 - d$ , then there are no  $\sigma$ -semistable framed Hitchin pairs of type  $(d, 2, L, \mathcal{O}_X(m_0), *)$ .

ii) If  $(E, \varepsilon, \varphi, \psi)$  is a  $\sigma$ -(semi)stable Hitchin pair of type  $(d, 2, L, \mathcal{O}_X(m_0), *)$ , then there exists a  $\sigma' \in \mathbb{Q}_{>0}$  such that  $(E, \psi)$  is a  $\sigma'$ -(semi)stable framed module in the sense of [1].

*Proof.* i) Indeed, if  $E$  is given as an extension  $(e)$  as before, then  $\sigma$ -semistability implies  $d - m_0 \leq (d - \sigma)/2$ , because  $\ker \psi$  is  $\varphi$ -invariant. We also see that we must choose  $m_0 > d/2$ .

ii) Again, let  $(E, \psi)$  be given by the extension  $(e)$ . We know  $d + \deg(D) - m_0 \leq (d - \sigma)/2$ . Set  $\sigma' := -d - 2\deg(D) + 2m_0$ . Then  $d + \deg(D) - m_0 = (d - \sigma')/2$ , and for any line subbundle  $F \neq \ker \psi$  of  $E$  we have  $\deg(F) \leq -\deg(D) + m_0 = (d + \sigma')/2$ . If the extension is non-split, then we can choose  $\sigma'$  slightly smaller, so that  $(E, \psi)$  is even  $\sigma'$ -stable.  $\square$

*Remark 1.22.* The analogous statement of 1.21 i) for framed Hitchin pairs of type  $(d, 2, L, \mathcal{H})$  is false. Indeed, consider, e.g.,  $X = \mathbb{P}_1$ ,  $L = \mathcal{H} = \mathcal{O}_{\mathbb{P}_1}$ ,  $E = \mathcal{O}_{\mathbb{P}_1}^{\oplus 2}$ ,  $\varepsilon := 1$ , let  $\psi$  be the projection onto the first factor and  $\varphi$  be given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then  $(E, \varepsilon, \varphi, \psi)$  is  $\sigma$ -stable for any  $\sigma > 0$ . We will come back to this in the section about boundedness in the next chapter.

*Example 1.23.* We look at the situation  $X = \mathbb{P}_1$ ,  $d = 0$ ,  $r = 2$ ,  $L = \mathcal{H} = \mathcal{O}_{\mathbb{P}_1}$ . Then, for a  $\sigma$ -semistable framed Hitchin pair  $(E, \varepsilon, \varphi, \psi)$ , we will have  $E \cong \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}$ , and  $\ker \psi$  can't be  $\varphi$ -invariant. Thus,  $(E, \varepsilon, \varphi, \psi)$  is  $\sigma$ -semistable if and only if  $\ker \psi$  is not  $\varphi$ -invariant and either  $\varepsilon \neq 0$  or  $\varphi^2 \neq 0$ . In particular,  $(E, \varepsilon, \varphi, \psi)$  is then stable for any  $\sigma > 0$ , and  $(E, \varepsilon, \varphi)$  is a semistable Hitchin pair. Denote by  $\mathcal{FH}$  the moduli space, and by  $\mathcal{H}$  the moduli space of semistable Hitchin pairs. There is a natural map  $\pi: \mathcal{FH} \rightarrow \mathcal{H}$ . The space  $\mathcal{H}$  is isomorphic to  $\mathbb{P}_2$  with coordinates, say,  $[l_0, l_1, l_2]$ . Here, the point  $[l_0, l_1, l_2]$  represents the class of the Hitchin pair  $(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}, \varepsilon, \varphi)$  with

$$\varepsilon = l_0, \quad \varphi = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}.$$

It is easy to describe the fibres of  $\pi$ : The preimage of  $[l_0, l_1, l_2]$  with  $l_1 \neq l_2$  consists just of the class of  $(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}, \varepsilon, \varphi, \psi)$  with

$$\varepsilon = l_0, \quad \varphi = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}, \quad \psi = (1, 1);$$

and the preimage of  $[l_0, l, l]$  of the class of  $(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}, \varepsilon, \varphi, \psi)$  with

$$\varepsilon = l_0, \quad \varphi = \begin{pmatrix} l & 1 \\ 0 & l \end{pmatrix}, \quad \psi = (1, 0).$$

Hence,  $\pi$  is an isomorphism. It is possible to give explicit coordinates for  $\mathcal{FH}$ . For this, we write  $\mathcal{FH} = \mathbb{P}(M_1^\vee \oplus M_2^\vee) // (\mathbb{C}^* \times \mathrm{SL}_2(\mathbb{C}))$ , where  $M_1 = \mathbb{C} \oplus \mathbb{C}^{(2,2)}$  and  $M_2 = \mathbb{C}^{2^\vee}$ , and  $\mathbb{C}^*$  acts with weights  $-1$  and  $2$ , so that the induced polarization on  $\mathbb{P}(M_1^\vee) \times \mathbb{P}(M_2^\vee)$  is  $\mathcal{O}(2, 1)$ . We choose coordinates  $(l_0, l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}, s_1, s_2)$ . Observe, that a  $(2 \times 2)$ -matrix is nilpotent if and only its determinant and its trace are both  $= 0$ . Moreover, “ $\ker \psi$  is  $\varphi$ -invariant” can be expressed as

$$\begin{aligned} D &:= \det \begin{pmatrix} l_{1,1}s_2 - l_{2,1}s_1 & s_2 \\ l_{2,1}s_2 - l_{2,2}s_1 & s_1 \end{pmatrix} \\ &= s_1s_2(l_{1,1} + l_{2,2}) - s_2^2l_{2,1} - s_1^2l_{1,2} = 0. \end{aligned}$$

Thus, the  $\mathrm{SL}_2(\mathbb{C})$ -nullforms are the common zeroes of the polynomials  $H_0 := l_0$ ,  $H_1 := l_{1,1} + l_{2,2}$ ,  $H_2 := l_{1,1}l_{2,2} - l_{1,2}l_{2,1}$ , and  $H_3 := D$ . We have to find those homogeneous polynomials in  $H_0, \dots, H_3$  which are  $\mathbb{C}^*$ -invariant. The weights of  $H_0, H_1, H_2$ , and  $H_3$  w.r.t. the  $\mathbb{C}^*$ -action are 1, 1, 2 and  $-2$ . Hence, the coordinates are given by  $h_0 = H_0^2 H_4$ ,  $h_1 = H_1^2 H_4$ , and  $h_3 = H_3 H_4$ .

## 2. ORIENTED FRAMED HITCHIN PAIRS

We will first discuss the notion of oriented framed Hitchin pairs and introduce the — parameter independent — semistability concept for them. Then we will proceed to construct the moduli spaces of semistable oriented framed Hitchin pairs over curves. There are some intricate technical points such as the behaviour of  $\sigma$ -semistability when  $\sigma$  becomes large.

**ORIENTED (SYMMETRIC) FRAMED HITCHIN PAIRS.** An *oriented framed Hitchin pair* of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  is a quintuple  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  where  $\mathcal{E}$ ,  $\varepsilon$ ,  $\varphi$ , and  $\psi$  have the same meaning as before, only that  $\psi = 0$  is now allowed, and  $\delta: \det \mathcal{E} \rightarrow \mathcal{N}[\mathcal{E}]$  is a homomorphism. An *isomorphism between oriented framed Hitchin pairs*  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  and  $(\mathcal{E}', \varepsilon', \delta', \varphi', \psi')$  of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  is an isomorphism  $\rho: \mathcal{E} \rightarrow \mathcal{E}'$  such that there exist numbers  $w, z \in \mathbb{C}^*$  such that

$$\varepsilon' = z\varepsilon, \delta' = w^r \delta \circ (\det \rho)^{-1}, \varphi' = z((\rho \otimes \mathrm{id}_{\mathcal{G}}) \circ \varphi \circ \rho^{-1}), \text{ \& } \psi' = w\psi \circ \rho^{-1}.$$

An isomorphism  $\rho$  as above will be called a *proper isomorphism*, if  $w = 1$ . Note that both notions of isomorphism yield the same equivalence relation on the set of all oriented framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$ . We will call an oriented framed Hitchin pair  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  *symmetric*, if  $\varphi$  is symmetric. As before, any symmetric oriented framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  defines an element  $\widehat{\chi}(\mathcal{E}, \varepsilon, \delta, \varphi, \psi) \in \widehat{\mathbb{P}}_{\mathcal{G}}$ . This depends only on its equivalence class and is called the *characteristic polynomial* of  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$ . In general, we can assign to every oriented framed Hitchin pair  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  its *characteristic vector* in  $\widehat{\Xi}_{\mathcal{G}}$ .

*Remark 2.1.* i) The automorphisms of the oriented framed Hitchin pairs living in the universal family which will be constructed below coming from actions of the stabilizers are only automorphisms in the weaker sense. Thus, one should adopt this notion of isomorphism in order to avoid confusion when trying to make the universal family descend. The above notion of isomorphism is also the one which extends to families. This situation is unlike the situation for framed Hitchin pairs! Note that  $\lambda \cdot \mathrm{id}$ ,  $\lambda \in \mathbb{C}^*$ , is always an automorphism. Moreover, if  $\rho$  is an isomorphism with constant  $w$ , then  $w^{-1}\rho$  will be a proper isomorphism.

ii) If  $X$  is a curve and  $L = \mathcal{G}$  is a line bundle, we can also extend the definitions of Chapter 1, i.e., we can define the notion of an *oriented framed Hitchin pair of type*  $(d, r, L, \mathcal{H}, \mathcal{N}, *)$ , of type  $(M, r, L, \mathcal{H})$  and  $(M, r, L, \mathcal{H}, *)$ . In the latter two cases we mean  $\det(E) \cong M$ , and  $\delta: \det(E) \rightarrow M$ .

A *family of oriented framed Hitchin pairs of type*  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  parametrized by the *noetherian scheme*  $S$  is defined to be a septuple  $(\mathcal{E}_S, \varepsilon_S, \delta_S, \varphi_S, \widehat{\psi}_S, \mathfrak{M}_S, \mathfrak{N}_S)$  consisting of line bundles  $\mathfrak{M}_S$  and  $\mathfrak{N}_S$  on  $S$ , an  $S$ -flat family of torsion free coherent sheaves  $\mathcal{E}_S$  with Hilbert polynomial  $P$  on  $S \times X$ , a section  $\varepsilon_S \in H^0(\mathfrak{N}_S)$ , a homomorphism  $\delta_S: \det(\mathcal{E}_S) \rightarrow \mathcal{N}[\mathcal{E}_S] \otimes \pi_S^* \mathfrak{M}_S$ , a twisted endomorphism  $\varphi_S: \mathcal{E}_S \rightarrow \mathcal{E}_S \otimes \pi_S^* \mathfrak{N}_S \otimes \pi_X^* \mathcal{G}$ , and a homomorphism  $\widehat{\psi}: S^r \mathcal{E}_S \rightarrow \pi_X^* S^r \mathcal{H} \otimes \pi_S^* \mathfrak{M}_S$  which is outside the closed subscheme  $S_0 := \{s \in S \mid \widehat{\psi}_{S|_{\{s\} \times X}} = 0\}$  a symmetric power, i.e., there is a line bundle  $\mathfrak{M}'$  on  $S \setminus S_0$  with  $\mathfrak{M}'^{\otimes r} = \mathfrak{M}_{S|_{S \setminus S_0}}$ , and  $\widehat{\psi}_{S|_{S \setminus S_0}}$  is the symmetric power of a homomorphism  $\psi': \mathcal{E}_{S|(S \setminus S_0) \times X} \rightarrow \pi_X^* \mathcal{H} \otimes \pi_{S \setminus S_0}^* \mathfrak{M}'$ . We leave it to the reader to define *equivalence of families*.

**SEMISTABILITY.** The definition of semistability will be made in analogy to the definition of semistability for oriented pairs in [8]. We need a preparatorial result.

**LEMMA 2.2.** *Let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be a framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$ , possibly with  $\psi = 0$ . Suppose there are non-trivial  $\varphi$ -invariant subsheaves in  $\ker \psi$ . Then there exists a uniquely determined non-trivial  $\varphi$ -invariant subsheaf  $\mathcal{K}_{\max} \subset \ker \psi$ , such that for all other  $\varphi$ -invariant subsheaves  $\mathcal{K} \subset \ker \psi$ , one has  $(P(\mathcal{K})/\mathrm{rk} \mathcal{K}) \leq (P(\mathcal{K}_{\max})/\mathrm{rk} \mathcal{K}_{\max})$ , and, if equality occurs,  $\mathcal{K} \subset \mathcal{K}_{\max}$ .*

*Proof.* Indeed, since, by assumption, the set of non-trivial  $\varphi$ -invariant subsheaves in  $\ker \psi$  is not empty and the sum and the intersection of two  $\varphi$ -invariant subsheaves in  $\ker \psi$  is again a  $\varphi$ -invariant subsheaf in  $\ker \psi$ , to get the result, one merely needs to copy the proof of Lemma 1.3.5 in [2].  $\square$

Set  $\sigma_{\mathcal{E}, \varphi, \psi} := P(\mathcal{E}) - (\mathrm{rk} \mathcal{E} / \mathrm{rk} \mathcal{K}_{\max}) P(\mathcal{K}_{\max})$ . Let  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  be an oriented framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$ . We call it *semistable*, if and only if either there are no  $\varphi$ -invariant subsheaves in  $\ker \psi$ , or  $\delta$  is an isomorphism and there are  $\varphi$ -invariant subsheaves in  $\ker \psi$ ,  $\sigma_{\mathcal{E}, \varphi, \psi} \geq 0$  and for all non-trivial  $\varphi$ -invariant subsheaves  $\mathcal{F} \subset \mathcal{E}$

$$\frac{P_{\mathcal{F}}}{\mathrm{rk} \mathcal{F}} - \frac{\sigma_{\mathcal{E}, \varphi, \psi}}{\mathrm{rk} \mathcal{F}} \leq \frac{P_{\mathcal{E}}}{\mathrm{rk} \mathcal{E}} - \frac{\sigma_{\mathcal{E}, \varphi, \psi}}{\mathrm{rk} \mathcal{E}}.$$

And we call  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  *stable*, if and only if either there are no  $\varphi$ -invariant subsheaves in  $\ker \psi$ , or  $\delta$  is an isomorphism and there are  $\varphi$ -invariant subsheaves in  $\ker \psi$ ,  $\sigma_{\mathcal{E}, \varphi, \psi} > 0$ , and one of the following two possibilities holds:

1. For all non-trivial  $\varphi$ -invariant proper subsheaves  $\mathcal{F} \subset \mathcal{E}$

$$\frac{P_{\mathcal{F}}}{\mathrm{rk} \mathcal{F}} - \frac{\sigma_{\mathcal{E}, \varphi, \psi}}{\mathrm{rk} \mathcal{F}} < \frac{P_{\mathcal{E}}}{\mathrm{rk} \mathcal{E}} - \frac{\sigma_{\mathcal{E}, \varphi, \psi}}{\mathrm{rk} \mathcal{E}}.$$

2.  $\psi \neq 0$ ,  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  splits as  $(\mathcal{K}_{\max}, \varepsilon, \varphi|_{\mathcal{K}_{\max}}, 0) \oplus (\mathcal{E}', \varepsilon, \varphi|_{\mathcal{E}'}, \psi)$ , the triple  $(\mathcal{K}_{\max}, \varepsilon, \varphi|_{\mathcal{K}_{\max}})$  is a stable Hitchin pair, and  $(\mathcal{E}', \varepsilon, \varphi|_{\mathcal{E}'}, \psi)$  is a  $\sigma_{\mathcal{E}, \varphi, \psi}$ -stable framed Hitchin pair, such that  $P(\mathcal{K}_{\max})/\mathrm{rk} \mathcal{K}_{\max} = (P(\mathcal{E}') - \sigma_{\mathcal{E}, \varphi, \psi})/\mathrm{rk} \mathcal{E}'$ .

For our constructions, we have to restate the semistability concept in terms of the semistability concepts for framed Hitchin pairs of Chapter 1.

**LEMMA 2.3.** i) *An oriented framed Hitchin pair  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  is semistable, if and only if it satisfies one of the following three conditions:*

1. *There are no  $\varphi$ -invariant subsheaves in the kernel of  $\psi$ .*
2.  *$\delta$  is an isomorphism, and  $(\mathcal{E}, \varepsilon, \varphi)$  is a semistable Hitchin pair of type  $(P, \mathcal{G})$ .*
3.  *$\psi \neq 0$ ,  $\delta$  is an isomorphism, and there is a polynomial  $\sigma \in \mathbb{Q}[t]$  of degree less than  $\dim X$  with positive leading coefficient s. th.  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is a  $\sigma$ -semistable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$ .*

ii)  *$(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  is stable, if and only if satisfies one of the conditions listed below.*

1. *There are no  $\varphi$ -invariant subsheaves in  $\ker \psi$ .*
2.  *$\delta$  is an isomorphism, and  $(\mathcal{E}, \varepsilon, \varphi)$  is a stable Hitchin pair of type  $(P, \mathcal{G})$ .*
3.  *$\psi \neq 0$ ,  $\delta$  is an isomorphism, and there is a polynomial  $\sigma \in \mathbb{Q}[t]$  of degree less than  $\dim X$  with positive leading coefficient s. th.  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is a  $\sigma$ -stable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$ .*
4.  *$\psi \neq 0$ ,  $\delta$  is an isomorphism, and there is a polynomial  $\sigma \in \mathbb{Q}[t]$  of degree less than  $\dim X$  with positive leading coefficient s. th.  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is a  $\sigma$ -polystable framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  of the form  $(\mathcal{E}', \varepsilon, \varphi', 0) \oplus (\mathcal{E}'', \varepsilon, \varphi'', \psi)$ .*

**Remark 2.4.** In the case the base  $X$  is a curve,  $L = \mathcal{G}$  a line bundle, and  $r = 2$ , then, by Lemma 1.21, for any (semi)stable oriented framed Hitchin pair  $(E, \varepsilon, \delta, \varphi, \psi)$  of type  $(d, 2, L, \mathcal{H}, \mathcal{N}, *)$ , then either  $(E, \varepsilon, \varphi)$  is a (semi)stable Hitchin pair of type  $(d, 2, L)$ , or the triple  $(E, \delta, \psi)$  is a (semi)stable oriented pair of type  $(d, 2, \mathcal{N})$  in the sense of [8].

Let  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  be a semistable oriented framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  which is not stable. This occurs if and only if there is either a  $\varphi$ -invariant subsheaf in  $\ker \psi$  which destabilizes  $(\mathcal{E}, \varepsilon, \varphi)$  as a Hitchin pair — in which case  $(\mathcal{E}, \varepsilon, \varphi)$  must be a semistable Hitchin pair — or there are a  $\varphi$ -invariant subsheaf  $\mathcal{K}$  of  $\ker \psi$  and a  $\varphi$ -invariant subsheaf  $\mathcal{F} \not\subset \ker \psi$  with

$$\frac{P_{\mathcal{F}}}{\operatorname{rk} \mathcal{F}} - \frac{\sigma_{\mathcal{K}}}{\operatorname{rk} \mathcal{F}} = \frac{P_{\mathcal{E}}}{\operatorname{rk} \mathcal{E}} - \frac{\sigma_{\mathcal{K}}}{\operatorname{rk} \mathcal{E}},$$

where  $\sigma_{\mathcal{K}} := P(\mathcal{E}) - (\operatorname{rk} \mathcal{E} / \operatorname{rk} \mathcal{K})P(\mathcal{K})$ . Note that in this case  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is  $\sigma_{\mathcal{K}}$ -semistable but not  $\sigma$ -semistable for every polynomial  $\sigma \neq \sigma_{\mathcal{K}}$ . Let  $0 \subset \mathcal{E}_m \subset \dots \subset \mathcal{E}_1 \subset \mathcal{E}$  be either the Jordan-Hölder filtration of the semistable Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi)$  or the Jordan-Hölder filtration of the  $\sigma_{\mathcal{K}}$ -semistable framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$ . Note that such a filtration induces a canonical isomorphism between  $\det(\bigoplus_{i=1}^{m+1} \mathcal{E}_{i-1} / \mathcal{E}_i)$  and  $\det(\mathcal{E})$ . Hence, we obtain an associated graded object  $\operatorname{gr}(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  — well-defined up to equivalence —, and we call  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  *polystable*, if it is equivalent to its associated graded object. Furthermore, two semistable oriented framed Hitchin pairs are said to be *S-equivalent*, if and only if their associated graded objects are equivalent.

*Remark 2.5.* i) At this moment, one could get the impression that a stable oriented framed Hitchin pair  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  where  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  splits into two components  $(\mathcal{E}', \varepsilon, \varphi', 0)$  and  $(\mathcal{E}'', \varepsilon, \varphi'', \psi)$  might not be stable at all, because there might be another semistable oriented framed Hitchin pair degenerating to it. This is, of course, not the case. Let  $\sigma_0$  be the unique polynomial w.r.t. which  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is semistable. Suppose  $(\tilde{\mathcal{E}}, \varepsilon, \tilde{\varphi}, \tilde{\psi})$  is a  $\sigma_0$ -semistable pair whose associated graded object is equivalent to  $(\mathcal{E}, \varepsilon, \varphi, \psi)$ . Then it is easy to see that either it is itself equivalent to  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  or it is  $\sigma$ -stable w.r.t. some polynomial  $\sigma$  which is "close" to  $\sigma_0$ .

ii) If  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  is properly semistable and  $(\mathcal{E}, \varepsilon, \varphi)$  is not a semistable Hitchin pair, then  $m$  in the above Jordan-Hölder filtration must be at least two.

iii) One checks that the stable oriented framed Hitchin pairs are precisely the polystable oriented framed Hitchin pairs which have only finitely many proper automorphisms.

**BOUNDEDNESS.** In this section, we prove the boundedness of the set of isomorphism classes of torsion free coherent sheaves occurring in semistable oriented framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  and carefully examine how the notion of  $\sigma$ -semistability behaves when  $\sigma$  becomes in a certain sense large. Invoking Maruyama's boundedness result [6] again, the boundedness will follow from

**PROPOSITION 2.6.** *Suppose  $\mathcal{G} = \mathcal{O}_X(m)^{\oplus u}$  and  $\mathcal{O}_X(m)$  is globally generated. Then, there is a constant  $C$  such that, for any semistable oriented framed Hitchin pair  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$ , the condition*

$$\mu_{\max}(\mathcal{E}) \leq C$$

*holds true.*

*Proof.* For  $\psi = 0$ , this is Proposition 2.2.2 in [8]. Thus, we can assume  $\psi \neq 0$ . Any subsheaf  $\mathcal{F}$  of  $\mathcal{E}$  can be written as an extension

$$0 \longrightarrow \ker \psi \cap \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \psi(\mathcal{F}) \longrightarrow 0.$$

Since  $\mu(\psi(\mathcal{F}))$  is bounded by  $\mu_{\max}(\mathcal{H})$ , it suffices to bound  $\mu_{\max}(\ker \psi)$ . Recall the following

**LEMMA 2.7.** *Given torsion free coherent sheaves  $\mathcal{F}_1$  and  $\mathcal{F}_2$  with  $\mu_{\min}(\mathcal{F}_1) > \mu_{\max}(\mathcal{F}_2)$ . Then there does not exist any non-trivial homomorphism from  $\mathcal{F}_1$  to  $\mathcal{F}_2$ .*

Set  $\mu_0 := \max\{\mu, \mu_{\max}(\mathcal{H})\}$ . We will derive a contradiction from the following assumption:

$$\mu_{\max}(\ker \psi) > \mu_0 + r \deg \mathcal{O}_X(m).$$



Suppose this was true and let  $0 = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_\kappa = \ker \psi$  be the slope Harder-Narasimhan filtration.

CLAIM 1. *For  $i = 1, \dots, \kappa - 1$ , the following inequality is satisfied:*

$$\mu(\mathcal{K}_i / \mathcal{K}_{i-1}) \leq \mu(\mathcal{K}_{i+1} / \mathcal{K}_i) + \deg \mathcal{O}_X(m),$$

in particular,

$$\mu_{\max}(\ker \psi) \leq \mu_{\min}(\ker \psi) + (r-1) \deg \mathcal{O}_X(m).$$

Therefore,  $\mu(\ker \psi) > \mu$ , so that  $\ker \psi$  cannot be  $\varphi$ -invariant. On the other hand,

$$\begin{aligned} \mu_{\min}(\ker \psi) &> \mu_{\max}(\mathcal{H}) + \deg \mathcal{O}_X(m) = \mu_{\max}(\mathcal{H} \otimes \mathcal{O}_X(m)^{\oplus u}) \\ &\geq \mu_{\max}((\mathcal{E} / \ker \psi) \otimes \mathcal{O}_X(m)^{\oplus u}). \end{aligned}$$

In view of Lemma 2.7, this means that  $\ker \psi$  must be  $\varphi$ -invariant.

To see the claim, first note that  $\varphi(\mathcal{K}_1) \subset \ker \psi \otimes \mathcal{O}_X(m)^{\oplus u}$ , again by 2.7. By semistability,  $\mathcal{K}_1$  cannot be  $\varphi$ -invariant. Hence, there is an index  $i' > 1$ , s. th.  $\varphi(\mathcal{K}_1) \subset \mathcal{K}_{i'} \otimes \mathcal{O}_X(m)^{\oplus u}$  and  $\varphi(\mathcal{K}_1) \not\subset \mathcal{K}_{i'-1} \otimes \mathcal{O}_X(m)^{\oplus u}$ . Thus, there is a non-trivial homomorphism  $f: \mathcal{K}_1 \rightarrow (\mathcal{K}_{i'} / \mathcal{K}_{i'-1}) \otimes \mathcal{O}_X(m)^{\oplus u}$ , whence

$$\mu(\mathcal{K}_1) \leq \mu(\mathcal{K}_{i'} / \mathcal{K}_{i'-1}) + \deg \mathcal{O}_X(m) \leq \mu(\mathcal{K}_1 / \mathcal{K}_2) + \deg \mathcal{O}_X(m).$$

Next, suppose the claim is true for  $i = 1, \dots, j$ . Since  $\mu_{\min}(\mathcal{K}_{j+1}) = \mu(\mathcal{K}_{j+1} / \mathcal{K}_j) \geq \mu(\mathcal{K}_1) - j \deg \mathcal{O}_X(m)$ , again  $\varphi(\mathcal{K}_{j+1}) \subset \ker \psi \otimes \mathcal{O}_X(m)^{\oplus u}$ , so that the same argumentation as before goes through, and we settle the case  $j+1$ .  $\square$

COROLLARY 2.8. *Let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be a framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$ , such that there is no  $\varphi$ -invariant subsheaf which is contained in  $\ker \psi$ , then  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  will be  $\sigma$ -stable for all polynomials  $\sigma \in \mathbb{Q}[t]$  of degree  $\dim X - 1$  with sufficiently positive leading coefficient.*

*Proof.* In the above proof, we have ruled out that one of the  $\mathcal{K}_i$  be  $\varphi$ -invariant by the semistability condition, here, we do it by assumption. Thus, the same conclusion as in Theorem 2.6 — with the same constant  $C$  — holds for framed Hitchin pairs  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  with non-trivial framing and no  $\varphi$ -invariant subsheaves in  $\ker \psi$ . So, any polynomial  $\sigma \in \mathbb{Q}[t]$  of degree  $\dim X - 1$  with leading coefficient  $> r(r-1)C - (r-1)d$  will do the trick.  $\square$

We also have the converse

PROPOSITION 2.9. *For all polynomials  $\sigma$  of degree  $\dim X - 1$  whose leading coefficient is sufficiently large and all  $\sigma$ -semistable framed Hitchin pairs  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$ , there will be no  $\varphi$ -invariant subsheaf in  $\ker \psi$ .*

*Proof.* Let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be a framed Hitchin pair, set  $\mathcal{F}_0 := \ker \psi$ , and for  $i \geq 1$ ,  $\mathcal{F}_i := \ker(\mathcal{F}_{i-1} \rightarrow (\mathcal{E} / \mathcal{F}_{i-1}) \otimes \mathcal{O}_X(m)^{\oplus u})$ . This yields a decreasing chain of saturated submodules

$$0 \subset \cdots \subset \mathcal{F}_{i+1} \subset \mathcal{F}_i \subset \cdots \subset \mathcal{F}_0.$$

By definition, a  $\varphi$ -invariant subsheaf  $\mathcal{F} \subset \ker \psi$  is contained in all the  $\mathcal{F}_i$ . Therefore, such a subsheaf exists if and only if one of the  $\mathcal{F}_i$  is  $\varphi$ -invariant. But, by construction, the  $\mathcal{F}_i$ 's coming from framed Hitchin pairs which are  $\sigma$ -semistable for some polynomial  $\sigma$  form bounded families. This means, if  $[d - (\text{leading coefficient of } \sigma)]/r$  is smaller than every possible  $\mu(\mathcal{F}_i)$ , then a  $\sigma$ -semistable framed Hitchin pair  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$  has no non-trivial  $\varphi$ -invariant subsheaves which are contained in  $\ker \psi$ .  $\square$

FLIPS BETWEEN THE MODULI SPACES OF FRAMED HITCHIN PAIRS. We will now carry out a discussion of this topic which makes the phenomenon completely transparent, and which is independent of the existence of master spaces.

Let  $\mathbb{Q}[t]_{\dim X-1,+}$  be the set of all polynomials of degree at most  $\dim X - 1$  which have positive leading coefficient. This set is totally ordered by the lexicographic order. Let  $\mathfrak{O}FH_{P/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{ss}$  be the set of equivalence classes of semistable oriented framed Hitchin pairs of type  $(P, \mathcal{G}, \mathcal{H}, \mathcal{N})$ . Given the equivalence class of a semistable framed oriented Hitchin pair  $(\mathcal{E}, \varepsilon, \delta, \varphi, \psi)$ , then a non-trivial  $\varphi$ -invariant saturated subsheaf  $\mathcal{K} \subset \ker \psi$  of  $\mathcal{E}$  defines a polynomial  $\sigma_{\mathcal{K}} \in \mathbb{Q}[t]_{\dim X-1,+}$ . Let  $\mathcal{Q}_{\text{dest}}$  be the subset of  $\mathbb{Q}[t]_{\dim X-1,+}$  of polynomials arising in that way. Pick a polynomial  $\sigma_{\infty}$  for which the conclusion of Proposition 2.9 holds.

LEMMA 2.10. *The set  $\mathcal{Q}_{\text{dest}} \cap \{\sigma \in \mathbb{Q}[t]_{\dim X-1,+} \mid \sigma \leq \sigma_{\infty}\}$  is finite.*

*Proof.* The assumption  $\sigma_{\mathcal{K}} \leq \sigma_{\infty}$  provides a lower bound for  $\mu(\mathcal{K})$ . Since the possible coherent sheaves  $\mathcal{E}$  vary in a bounded family, by Proposition 2.6, i.e., they are all quotients of the sheaf  $\mathcal{O}_X(-n)^{\oplus v}$ , for some large  $n$  and  $v$ , the lemma follows from a result of Grothendieck's ([2], Lemma 1.7.9).  $\square$

Let  $\sigma'_1 < \dots < \sigma'_t$  be the polynomials in  $\mathcal{Q}_{\text{dest}}$  which are smaller than  $\sigma_{\infty}$ . This gives rise to "open intervals"  $I_0 := \{\sigma \mid \sigma < \sigma'_1\}$ ,  $I_i := \{\sigma \mid \sigma'_i < \sigma < \sigma'_{i+1}\}$ ,  $i = 1, \dots, t-1$ , and  $I_t := \{\sigma \mid \sigma'_t < \sigma\}$ .

LEMMA 2.11. *Let  $\sigma_1$  and  $\sigma_2$  be two polynomials in  $\mathbb{Q}[t]_{\dim X-1,+} \setminus \mathcal{Q}_{\text{dest}}$ . If  $\sigma_1$  and  $\sigma_2$  lie both in one of the  $I_i$ , then  $\mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma_1-ss} \cong \mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma_2-ss}$ .*

*Proof.* By the assumption that  $\sigma_1$  and  $\sigma_2$  do not lie in  $\mathcal{Q}_{\text{dest}}$ , we have  $\mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma_i-ss} = \mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma_i-s}$ ,  $i = 1, 2$ . Therefore, it is enough to show that a framed Hitchin pair of type  $(P, \mathcal{G}, \mathcal{H})$  is  $\sigma_1$ -stable if and only if it is  $\sigma_2$ -stable. First, let  $i < t$ . Suppose  $\sigma_1 < \sigma_2$ . If  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is  $\sigma_1$ -stable but not  $\sigma_2$ -stable, then there must be a  $\varphi$ -invariant subsheaf  $\mathcal{K} \subset \ker \psi$  with  $\sigma_1 < \sigma_{\mathcal{K}} < \sigma_2$ . The saturation of  $\mathcal{K}$  has the same property. But since there is no such polynomial, by assumption,  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  is also  $\sigma_2$ -stable. Next, let  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  be  $\sigma_2$ -stable. If it was not  $\sigma_1$ -stable, there would be a saturated  $\varphi$ -invariant subsheaf  $\mathcal{F}$  which is not contained in  $\ker \psi$  which  $\sigma_1$ -destabilizes  $(\mathcal{E}, \varepsilon, \varphi, \psi)$ . Define  $\sigma_{\mathcal{F}} \in \mathbb{Q}[t]_{\dim X-1,+}$  by the condition

$$\frac{P_{\mathcal{F}}}{\text{rk } \mathcal{F}} - \frac{\sigma_{\mathcal{F}}}{\text{rk } \mathcal{F}} = \frac{P_{\mathcal{E}}}{\text{rk } \mathcal{E}} - \frac{\sigma_{\mathcal{F}}}{\text{rk } \mathcal{E}}.$$

Then  $\sigma_1 < \sigma_{\mathcal{F}} < \sigma_2$ . If we choose  $\mathcal{F}$  such that  $\sigma_{\mathcal{F}}$  becomes maximal, then  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  will be properly  $\sigma_{\mathcal{F}}$ -semistable. Its associated graded object possesses a  $\varphi$ -invariant saturated subsheaf  $\mathcal{K} \subset \ker \bar{\psi}$  with  $\sigma_{\mathcal{K}} = \sigma_{\mathcal{F}}$ . This is again an impossibility. In the remaining case  $i = t$ , we may assume that either  $\sigma_1$  or  $\sigma_2$  agrees with  $\sigma_{\infty}$ . In the former case, i.e.,  $\sigma_{\infty} < \sigma_2$ , the assertion follows from Proposition 2.9. If  $\sigma_1 < \sigma_{\infty}$ , then the same argumentation as before can be applied.  $\square$

Now, pick for each  $i \in \{0, \dots, t\}$  a polynomial  $\sigma_i \in I_i$ . Observe that every  $(\mathcal{E}, \varepsilon, \varphi, \psi)$  which is  $\sigma_i$ -stable is also  $\sigma'_{i-1}$ - and  $\sigma'_i$ -semistable. Therefore, we obtain a diagram

$$\begin{array}{ccccc} \mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma_0-s} & & & & \mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma_t-s} \\ \swarrow & & \searrow & \dots & \swarrow \quad \searrow \\ \mathcal{H}_{P/\mathcal{G}}^{ss} & \mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma'_1-ss} & & & \mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma'_t-ss} \quad \mathcal{F}\mathcal{H}_{P/\mathcal{G}/\mathcal{H}}^{\sigma_t-s} \end{array}$$

Here,  $\mathcal{H}_{P/\mathcal{G}}^{ss}$  is the moduli space of semistable Hitchin pairs.

THE PARAMETER SPACE AND THE GROUP ACTIONS. To avoid further technicalities, we will from now on assume that  $X$  is a curve.

*Assumptions 2.12.* For any  $n \geq n_2$ , and any semistable oriented framed Hitchin pair  $(E, \varepsilon, \delta, \varphi, \psi)$  of type  $(d, r, \mathcal{G}, \mathcal{H}, \mathcal{N})$ :

- $\mathcal{H}(n)$  is globally generated.
- $E(n)$  is globally generated, and the first cohomology group of  $E(n)$  vanishes.
- The conclusion of Proposition 1.5 holds for all positive rational numbers  $\sigma$ , and also the analogous assertion for semistable Hitchin pairs of type  $(d, r, \mathcal{G})$ .
- Fix a positive rational number  $\sigma_\infty$ , for which the conclusion of Proposition 2.9 holds. Then,  $[d + r(n + 1 - g)]/2 \geq \sigma_\infty$ .

Again,  $n_2 = 0$  is assumed. We also adopt 1.13. We start as in the construction of the parameter space for framed Hitchin pairs. As explained in the Preliminaries, the universal quotient  $q_{\mathfrak{P}}: V \otimes \mathcal{O}_{\mathfrak{P} \times X} \rightarrow \mathfrak{E}_{\mathfrak{P}}$  defines a morphism  $d(\mathfrak{E}_{\mathfrak{P}}): \mathfrak{P} \rightarrow \text{Pic} X$ . By the universal property of the Picard scheme,  $\mathcal{D}_{\mathfrak{P}} := \pi_{\mathfrak{P}*}(\underline{\text{Hom}}(\det(\mathfrak{E}_{\mathfrak{P}}), \mathcal{N}[\mathfrak{E}_{\mathfrak{P}}]))$  is invertible, and for any point  $\mathbf{p} \in \mathfrak{P}$ ,

$$\mathcal{D}_{\mathfrak{P}}(\mathbf{p}) \cong \text{Hom}(\det(\mathfrak{E}_{\mathfrak{P}|_{\{\mathbf{p}\} \times X}}), \mathcal{N}[\mathfrak{E}_{\mathfrak{P}|_{\{\mathbf{p}\} \times X}]}).$$

Set  $\widehat{\mathfrak{T}} := \mathbb{P}(\mathcal{D}_{\mathfrak{P}}^\vee \oplus S^r \text{Hom}(V, H^0(\mathcal{H}))^\vee \otimes \mathcal{O}_{\mathfrak{P}})$ . Then, we can construct our parameter space as a closed subscheme  $\mathfrak{T}$  of  $\widehat{\mathfrak{T}}$ . Note that, outside the closed subscheme  $\mathbb{P}(\mathcal{D}_{\mathfrak{P}}^\vee)$  of  $\widehat{\mathfrak{T}}$ , we can extract the  $r$ -th root of the tautological line bundle. From this, it is clear that there is a universal family  $(\mathfrak{E}_{\mathfrak{T}}, \varepsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}}, \varphi_{\mathfrak{T}}, \widehat{\psi}_{\mathfrak{T}}, \mathfrak{M}_{\mathfrak{T}}, \mathfrak{N}_{\mathfrak{T}})$  on  $\mathfrak{T} \times X$  which has the local universal property.  $\mathfrak{T}_0$  is the open subscheme mapping to  $\Omega_0$ , and  $\mathfrak{T}_0^{(s)s}$  is the open subscheme parametrizing the (semi)stable oriented framed Hitchin pairs.

*Remark 2.13.* Since  $X$  curve, the quasi-projective scheme  $\Omega_0$  is smooth. Therefore, the restriction of the universal quotient to  $\Omega_0 \times X$  is locally free of rank  $r$ .

There is a natural right action by  $\text{SL}(V)$  on  $\mathfrak{T}$ , and the universal family on  $\mathfrak{T} \times X$  comes again with an  $\text{SL}(V)$ -linearization. Remark 2.1 i) and 2.5 iii) show that all the stabilizers of points in  $\mathfrak{T}_0$  which are represented by stable oriented framed Hitchin pairs are indeed finite. This is because there are only finitely multiples of  $\text{id}_V$  in  $\text{SL}(V)$ . We must construct the good (geometric) quotient  $\mathfrak{T}_0^{(s)s} // \text{SL}(V)$ .

*The Gieseker map.* We let  $\mathcal{J} \subset \text{Pic} X$  be the Jacobian of degree  $d$  line bundles on  $X$  and  $\mathcal{N}_{\mathcal{J}}$  be the restriction of  $\mathcal{N}$  to  $\mathcal{J} \times X$ . If  $d > 2g - 2$ , then  $\mathfrak{A}_{\mathcal{J}} := \pi_{\mathcal{J}*} \mathcal{N}_{\mathcal{J}}$  and  $\mathfrak{A}'_{\mathcal{J}} := \pi_{\mathcal{J}*}(\mathcal{N}_{\mathcal{J}} \otimes \pi_X^* \mathcal{O}_X(m))^{\oplus u}$  are locally free. From the universal family, we get, on  $\mathfrak{T}_0 \times X$ , homomorphisms

$$\begin{aligned} \mathfrak{E}_{\mathfrak{T}_0} \otimes \mathfrak{E}_{\mathfrak{T}_0}^\vee &\longrightarrow \pi_{\mathfrak{T}_0}^* \mathfrak{N}_{\mathfrak{T}_0} \otimes \pi_X^* \mathcal{O}_X(m)^{\oplus u} \\ \mathcal{O}_{\mathfrak{T}_0 \times X} &\longrightarrow \pi_{\mathfrak{T}_0}^* \mathfrak{N}_{\mathfrak{T}_0}. \end{aligned}$$

Observe  $\mathfrak{E}_{\mathfrak{T}_0}^\vee \cong \bigwedge^{r-1} \mathfrak{E}_{\mathfrak{T}_0} \otimes \det(\mathfrak{E}_{\mathfrak{T}_0})^\vee$ , because  $\mathfrak{E}_{\mathfrak{T}_0}$  is locally free. Using the surjection  $V \otimes \mathcal{O}_{\mathfrak{T}_0 \times X} \rightarrow \mathfrak{E}_{\mathfrak{T}_0}$ , we obtain

$$\begin{aligned} V \otimes \bigwedge^{r-1} V \otimes \mathcal{O}_{\mathfrak{T}_0 \times X} &\longrightarrow \det(\mathfrak{E}_{\mathfrak{T}_0}) \otimes \pi_{\mathfrak{T}_0}^* \mathfrak{N}_{\mathfrak{T}_0} \otimes \pi_X^* \mathcal{O}_X(m)^{\oplus u} \\ \bigwedge^r V \otimes \pi_X^* \mathcal{O}_X(m)^{\oplus u} &\longrightarrow \det(\mathfrak{E}_{\mathfrak{T}_0}) \otimes \pi_{\mathfrak{T}_0}^* \mathfrak{N}_{\mathfrak{T}_0} \otimes \pi_X^* \mathcal{O}_X(m)^{\oplus u}. \end{aligned}$$

Now, project all this to  $\mathfrak{T}_0$ , so that you get

$$\begin{aligned} V \otimes \bigwedge^{r-1} V \otimes \mathcal{O}_{\mathfrak{T}_0} &\longrightarrow \mathfrak{A}'_{\mathfrak{T}_0} \otimes \mathfrak{N}_{\mathfrak{T}_0} \otimes \mathcal{L}_{\mathfrak{T}_0} \\ \bigwedge^r V \otimes M \otimes \mathcal{O}_{\mathfrak{T}_0} &\longrightarrow \mathfrak{A}'_{\mathfrak{T}_0} \otimes \mathfrak{N}_{\mathfrak{T}_0} \otimes \mathcal{L}_{\mathfrak{T}_0}. \end{aligned}$$

Here,  $\mathfrak{A}'_{\mathfrak{T}_0}$  is the pullback of  $\mathfrak{A}'_{\mathcal{J}}$  under the map  $d(\mathfrak{E}_{\mathfrak{T}_0})$ , and  $\mathcal{L}_{\mathfrak{T}_0}$  is some linearized line bundle. These data define an  $\mathrm{SL}(V)$ -equivariant morphism

$$\mathbf{t}_1: \mathfrak{T}_0 \longrightarrow \mathbf{P}_1 := \mathbb{P}(\mathrm{Hom}(\bigwedge^r V \otimes M \otimes \mathcal{O}_{\mathcal{J}} \oplus V \otimes \bigwedge^{r-1} V \otimes \mathcal{O}_{\mathcal{J}}, \mathfrak{A}'_{\mathcal{J}})^{\vee})$$

which factorizes over an injective morphism  $\mathfrak{P}_0 \longrightarrow \mathbf{P}_1$ , and  $\mathbf{t}_1^* \mathcal{O}_{\mathbf{P}_1}(1) = \mathfrak{N}_{\mathfrak{T}_0} \otimes \mathcal{L}_{\mathfrak{T}_0}$ . Next, we have a look at the data defined by the orientation and the framing, i.e., at

$$\begin{aligned} \bigwedge^r V \otimes \mathcal{O}_{\mathfrak{T}_0} &\longrightarrow \det(\mathfrak{E}_{\mathfrak{T}_0}) \longrightarrow \mathcal{N}[\mathfrak{E}_{\mathfrak{T}_0}] \otimes \pi_{\mathfrak{T}_0}^* \mathfrak{M}_{\mathfrak{T}_0} \\ S^r V \otimes \mathcal{O}_{\mathfrak{T}_0} &\longrightarrow S^r \mathfrak{E}_{\mathfrak{T}_0} \longrightarrow \pi_X^* S^r \mathcal{H} \otimes \pi_{\mathfrak{T}_0}^* \mathfrak{M}_{\mathfrak{T}_0}. \end{aligned}$$

Projecting these to  $\mathfrak{T}_0$ , provides us with

$$\begin{aligned} \bigwedge^r V \otimes \mathcal{O}_{\mathfrak{T}_0} &\longrightarrow \mathfrak{A}_{\mathfrak{T}_0} \otimes \mathfrak{M}_{\mathfrak{T}_0} \\ S^r \mathrm{Hom}(V, H^0(\mathcal{H})) \otimes \mathcal{O}_{\mathfrak{T}_0} &\longrightarrow \mathfrak{M}_{\mathfrak{T}_0}, \end{aligned}$$

and, thus, with a morphism

$$\mathbf{t}_2: \mathfrak{T}_0 \longrightarrow \mathbf{P}_2 := \mathbb{P}(\mathrm{Hom}(\bigwedge^r V \otimes \mathcal{O}_{\mathcal{J}}, \mathfrak{A}_{\mathcal{J}})^{\vee} \oplus S^r \mathrm{Hom}(V, H^0(\mathcal{H}))^{\vee} \otimes \mathcal{O}_{\mathcal{J}}),$$

such that  $\mathbf{t}_2^* \mathcal{O}_{\mathbf{P}_2}(1) = \mathfrak{M}_{\mathfrak{T}_0}$ . The resulting  $\mathrm{SL}(V)$ -equivariant and injective homomorphism

$$\mathbf{t}: \mathfrak{T}_0 \longrightarrow \mathbf{P}_1 \times_{\mathcal{J}} \mathbf{P}_2 \longrightarrow \mathbb{T} := \mathbf{P}_1 \times \mathbf{P}_2$$

is our Gieseker map. We linearize the  $\mathrm{SL}(V)$ -action on  $\mathbb{T}$  in a very ample line bundle of the form  $\mathcal{O}_{\mathbb{T}}(1, 1) \otimes (\text{pullback of a very ample line bundle on } \mathcal{J})$ .

*The semistable points in  $\mathbb{T}$ .* The key step to the construction of the moduli spaces is

**THEOREM 2.14.** *Let  $t = ([q: V \otimes \mathcal{O}_X \longrightarrow E], [\varepsilon, \varphi], [\delta, \psi])$  be a point in  $\mathfrak{T}_0$ . Then the associated point  $\mathbf{t}(r) \in \mathbb{T}$  is (semi/poly)stable w.r.t. the given linearization if and only if  $(E, \varepsilon, \delta, \varphi, \psi)$  is a (semi/poly)stable oriented framed Hitchin pair of type  $(d, r, \mathcal{G}, \mathcal{H}, \mathcal{N})$ .*

First, observe that the action on  $\mathcal{J}$  is trivial, so that  $\mathbf{t}(t)$  will be (semi)stable if and only if it is (semi)stable in  $\mathbb{T}_t := \mathbf{P}_{1,t} \times \mathbf{P}_{2,t}$  w.r.t. the linearization in  $\mathcal{O}(1, 1)$  where

$$\begin{aligned} \mathbf{P}_{1,t} &:= \mathbb{P}(\mathrm{Hom}(\bigwedge^r V \otimes M \oplus V \otimes \bigwedge^{r-1} V, H^0(\det(E)(m))^{\oplus u})^{\vee}); \\ \mathbf{P}_{2,t} &:= \mathbb{P}(\mathrm{Hom}(\bigwedge^r V, H^0(\det(E)))^{\vee} \oplus S^r \mathrm{Hom}(V, H^0(\mathcal{H}))^{\vee}). \end{aligned}$$

Next, we introduce on  $\mathbf{P}_{2,t}$  the  $\mathbb{C}^*$ -action which multiplies the second component by  $z$ . Then, there is a family of linearizations of the  $\mathbb{C}^*$ -action in  $\mathcal{O}_{\mathbf{P}_{2,t}}$  parametrized by natural numbers  $e, k$  with  $0 \leq e \leq k$  [8]. We look at the quotients of  $\mathbf{P}_{1,t} \times \mathbf{P}_{2,t}$  by these linearized  $\mathbb{C}^*$ -actions. If  $e = 0$ , then the quotient is  $\mathbf{T}_t^0 := \mathbf{P}_{1,t} \times \mathbb{P}(\mathrm{Hom}(\bigwedge^r V, H^0(\det(E)))^{\vee})$  with induced polarization  $\mathcal{O}(1, 1)$ . If  $e = k$ , then the quotient is

$$\mathbf{T}_t^{\infty} := \mathbf{P}_{1,t} \times \mathbb{P}(S^r \mathrm{Hom}(V, H^0(\mathcal{H}))^{\vee})$$

with induced polarization  $\mathcal{O}(1, 1)$ . In the other cases, the quotient is

$$\mathbf{T}_t := \mathbf{P}_{1,t} \times \mathbb{P}(\mathrm{Hom}(\bigwedge^r V, H^0(\det(E)))^{\vee}) \times \mathbb{P}(S^r \mathrm{Hom}(V, H^0(\mathcal{H}))^{\vee})$$

with induced polarization  $L_k^e := \mathcal{O}(k, k - e, e)$ . Define

$$\sigma_k^e := \frac{p}{2} \cdot \frac{e}{k}.$$

Note that, by 2.12, for a given (positive)  $\sigma < \sigma_{\infty}$ , we can find  $0 < e < k$  satisfying  $\sigma = (p/2)(e/k)$ . By the Preliminaries and Remark 2.5 iii), Theorem 2.14 now reduces to the following

THEOREM 2.15. i) *The associated point in  $\mathbf{T}_t^0$  is (semi/poly)stable if and only if  $(E, \varepsilon, \varphi)$  is a (semi/poly)stable Hitchin pair of type  $(d, r, \mathcal{G})$ .*

ii) *The associated by point in  $\mathbf{T}_t^\infty$  is (semi/poly)stable if and only if there are no  $\varphi$ -invariant subbundles  $F$  of  $E$  which are contained in the kernel of  $\psi$ .*

iii) *The associated point in  $\mathbf{T}_t$  is (semi/poly)stable w.r.t. the linearization  $L_k^e$  if and only if  $(E, \varepsilon, \varphi, \psi)$  is a  $\sigma_k^e$ -(semi/poly)stable framed Hitchin pair of type  $(d, r, \mathcal{G}, \mathcal{H})$ .*

*The  $\mathbb{P}(S^r \text{Hom}(V, H^0(\mathcal{H}))^\vee)$ -component.* By definition, the image of the point  $\mathbf{t}(t)$  in that space lies in the image of  $\mathbf{R}$  under the  $r$ -th Veronese map. Therefore, the weights are those in  $\mathbf{R}$  multiplied by  $r$ .

*The  $\mathbb{P}(\text{Hom}(\wedge^r V, H^0(\det(E))^\vee)$ -component.* Let  $v_1, \dots, v_p$  be a basis for  $V$ . Set  $E_i := q(\langle v_1, \dots, v_i \rangle \otimes \mathcal{O}_X)$ . For a one parameter subgroup  $\lambda$  of  $\text{SL}(V)$  which is given w.r.t. that basis by weights  $\gamma_1 \leq \dots \leq \gamma_p$ , one computes  $\mu([q], \lambda) = -\sum_{i=1}^p (\text{rk } E_i - \text{rk } E_{i-1}) \gamma_i$ , in particular,  $\mu([q], \lambda^{(i)}) = p \text{rk } E_i - ir$ .

*The  $\mathbf{P}_{1,t}$ -component.* Fix basess  $v_1, \dots, v_p$  of  $V$  and  $m_1, \dots, m_\mu$  of  $H^0(\mathcal{O}_X(m))$ , let  $m_1^1, \dots, m_\mu^1, \dots, m_\mu^u, \dots, m_\mu^u$  be the resulting basis for  $M$ , and  $\text{ev}: \wedge^r V \otimes M \longrightarrow H^0(\det(E)(m))$  is the natural map. Let's look at some special elements in the space  $\text{Hom}(\wedge^r V \otimes M \oplus V \otimes \wedge^{r-1} V, H^0(\det(E)(m)^{\oplus u}))$ . For each ordered set  $I$  of  $r$  elements in  $\{1, \dots, p\}$ , each  $k \in \{1, \dots, \mu\}$ , and  $l \in \{1, \dots, u\}$ , we define  $S_{I,k,l}$  as the element which maps  $(v_{i_1} \wedge \dots \wedge v_{i_r}) \otimes m_k^l$  to  $\text{ev}((v_{i_1} \wedge \dots \wedge v_{i_r}) \otimes m_k^l)$  and is zero on all other basis elements of  $\wedge^r V \otimes M$  and also zero on  $V \otimes \wedge^{r-1} V$ . This element is an eigenvector for the action of the maximal torus defined by  $v_1, \dots, v_p$ . Indeed, if  $\lambda$  is given by weights  $\gamma_1, \dots, \gamma_p$ , then it acts on  $S_{I,k,l}$  with weight  $\gamma_{i_1} + \dots + \gamma_{i_r}$ . In the same way, for  $I, k, l$  as before and  $i, j \in \{1, \dots, r\}$ , we define  $\Theta_{I,k,l}^{ij}$  as the element which maps  $v_{i_i} \otimes (v_{i_1} \wedge \dots \wedge \hat{v}_{i_j} \wedge \dots \wedge v_{i_r}) \otimes m_k^l$  to  $\text{ev}((v_{i_1} \wedge \dots \wedge v_{i_r}) \otimes m_k^l)$  and is zero on  $\wedge^r V \otimes M$  and all other basis vectors of  $V \otimes \wedge^{r-1} V$ . These are also eigenvectors, and  $\lambda$  as above acts with weight  $\gamma_{i_1} + \dots + \gamma_{i_r} - \gamma_{i_j} + \gamma_{i_i}$ . By definition, for any  $t \in \mathfrak{T}_0$ , the component  $\mathbf{t}_1(t)$  lies in the linear subspace of  $\mathbf{P}_{1,t}$  which is spanned by the  $S_{I,k,l}$  and  $\Theta_{I,k,l}^{ij}$ , and thus the computation of weights is analogous to that in the first part of this paper.

After these preparations, it is clear that i) and iii) in Theorem 2.15 can be proved in exactly the same way as Theorem 1.16 in the first part of this paper. In order to see also ii), we first observe that computations in the space  $\mathbf{T}_t^\infty$  give that the point  $\mathbf{t}_1(t)$  is (semi/poly)stable if and only if  $(E, \varepsilon, \varphi, \psi)$  is  $\sigma^*$ -(semi/poly)stable where  $\sigma^* = p/2$ . By Assumption 2.12,  $\sigma^* \geq \sigma_\infty$ , so that we can conclude by Proposition 2.9. Finally, a standard argument shows

PROPOSITION 2.16. *The map  $\mathbf{t}_{\mathfrak{T}_0^{ss}}: \mathfrak{T}_0^{ss} \longrightarrow \mathbb{T}^{ss}$  is a finite morphism.*

*The outcome.* The summary of the results of the previous paragraphs is given by

THEOREM 2.17. *The good quotient  $\mathfrak{T}_0^{ss} // \text{SL}(V)$  exists. It is a projective scheme, and the open subscheme  $\mathfrak{T}_0^s // \text{SL}(V)$  is a geometric quotient.*

THE MODULI SPACES. Let  $\text{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{(s)s}$  be the functor which assigns to each noetherian scheme  $S$  the set of equivalence classes of families of (semi)stable oriented framed Hitchin pairs of type  $(d, r, \mathcal{G}, \mathcal{H}, \mathcal{N})$  which are parametrized by  $S$ , and define the closed subfunctor  $\text{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}/\text{symm}}^{(s)s}$  of (semi)stable symmetric oriented framed Hitchin pairs. Finally, we set  $\mathcal{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{(s)s} := \mathfrak{T}_0^{(s)s} // \text{SL}(V)$ .

THEOREM 2.18. i) *There is a natural transformation  $\vartheta^{(s)s}$  of  $\text{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{(s)s}$  into the functor of points of  $\mathcal{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{(s)s}$  which is minimal in the usual sense (see i) of Thm. 1.7), so that  $\mathcal{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{(s)s}$  is a coarse moduli scheme for stable oriented framed Hitchin pairs of type  $(d, r, \mathcal{G}, \mathcal{H}, \mathcal{N})$ . The map  $\vartheta^{ss}(\mathbb{C})$  induces a bijection between the set of  $S$ -equivalence classes of semistable oriented framed Hitchin pairs of type*

$(d, r, \mathcal{G}, \mathcal{H}, \mathcal{N})$  and the set of closed points of  $\mathcal{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{ss}$ . There is also a proper generalized Hitchin map

$$\widehat{\xi}: \mathcal{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{ss} \longrightarrow \widehat{\Xi}_{\mathcal{G}}.$$

ii) There is a closed subscheme  $\mathcal{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}/\text{symm}}^{(s)s}$  of  $\mathcal{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}}^{(s)s}$  such that the analogues to i) w.r.t. the functor  $\text{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}/\text{symm}}^{(s)s}$  hold true. Furthermore, there is a Hitchin map

$$\widehat{\chi}: \mathcal{OFH}_{d/r/\mathcal{G}/\mathcal{H}/\mathcal{N}/\text{symm}}^{ss} \longrightarrow \widehat{\mathbb{P}}_{\mathcal{G}},$$

mapping a closed point of the moduli space to the characteristic polynomial of a represent-  
ing oriented framed Hitchin pair. The Hitchin map clearly is proper.

*Example 2.19.* We return to the setting of Example 1.23. Let  $\mathcal{OFH}$  be the master space. This time, we have to determine the quotient  $\mathbb{P}(M_1^\vee \oplus M_2^\vee \oplus \mathbb{C}) // (\mathbb{C}^* \times \text{SL}(V))$ . Denote the coordinates by  $(l_0, l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}, s_1, s_2, s_3)$ . The  $\text{SL}(V)$ -nullforms are cut out by the equations  $H_i = 0$ ,  $i = 0, \dots, 4$ , where  $H_0, H_1, H_2$ , and  $H_3$  are as before, and  $H_4 := s_3$ . Set  $g_0 := H_0^2$ ,  $g_1 := H_1^2$ , and  $g_2 := H_2$ . It follows easily that the master space  $\mathcal{OFH}$  is isomorphic to  $\mathbb{P}_2 \times \mathbb{P}_1$  with coordinates  $([g_0 : g_1 : g_2], [H_3 : H_4])$ . The Hitchin space  $\mathcal{H}$  is the  $\mathbb{C}^*$ -quotient of the open subset  $H_4 \neq 0$ , and the space  $\mathcal{FH}$  is the  $\mathbb{C}^*$ -quotient of the open subset  $H_3 \neq 0$ .

**THE  $(\mathbb{C}^* \times \mathbb{C}^*)$ -ACTION ON THE MODULI SPACE.** On  $\mathcal{OFH} := \mathcal{OFH}_{d/r/L/\mathcal{H}/\mathcal{N}}^{ss}$ , there are two  $\mathbb{C}^*$ -actions which commute with each other: First, there is the  $\mathbb{C}^*$ -action which comes from multiplying the twisted endomorphism by a scalar factor. Second, we can multiply the framing by a non-zero complex number, this yields the second  $\mathbb{C}^*$ -action. As explained in [8], it is important to study the fixed point sets of those  $\mathbb{C}^*$ -actions, the so-called *varieties of reductions*. The fixed point set of the first  $\mathbb{C}^*$ -action contains two obvious components. The first one is  $\mathcal{M}_{d/r/\mathcal{H}/\mathcal{N}}^{ss}$ , the master space of semistable oriented framed bundles as constructed in [8], corresponding to the points with  $\varphi = 0$ . The second one is  $\mathcal{OFH}_\infty := \mathcal{OFH}_{\neq 0} // \mathbb{C}^*$ , where  $\mathcal{OFH}_{\neq 0}$  is the open subset where  $\varepsilon \neq 0$ .

The fixed point set of the second  $\mathbb{C}^*$ -action looks as follows: First, there is  $\mathcal{H}_{d/r/\mathcal{G}}^{ss} = \{\psi = 0\}$ , the moduli space of semistable Hitchin pairs of type  $(d, r, \mathcal{G})$ . Second, there is  $\mathcal{FH}_{d/r/L/\mathcal{H}}^{\sigma_\infty - ss}$  embedded as the part  $\{\delta = 0\}$ . Third, there is the set of the stable points of the form  $((E', \varepsilon, \varphi', 0) \oplus (E'', \varepsilon, \varphi'', \psi), \delta)$  which has  $t$  components. From the GIT-process, these  $\mathbb{C}^*$ -actions come with natural linearizations in an ample line bundle on  $\mathcal{OFH}$ , let  $l$  be the one of the second  $\mathbb{C}^*$ -action, then, as in [8], one can now conclude

**THEOREM 2.20.** *For  $k > 0$  and  $e \in \mathbb{Z}$ , let  $l_k^e$  be the modification of the linearization  $l$  as described in Part I of [8]. Then the GIT-quotients  $\mathcal{OFH} //_{l_k^e} \mathbb{C}^*$  run through the moduli spaces  $\mathcal{H}_{d/r/\mathcal{G}}^{ss}$ , and  $\mathcal{FH}_{d/r/\mathcal{G}/\mathcal{H}}^{\sigma - ss}$ ,  $\sigma \in \mathbb{Q}_{>0}$ . In particular, the chain of flips described in a previous section is a chain of  $\mathbb{C}^*$ -flips.*

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## REFERENCES

- [1] D. Huybrechts and M. Lehn, *Framed modules and their moduli*. Int. J. Math. **6** (1995), 297-324
- [2] D. Huybrechts and M. Lehn, *The Geometry of the Moduli Spaces of Sheaves*. Vieweg, 1997
- [3] L. Le Bruyn and C. Procesi, *Semisimple representations of quivers*, Transactions Amer. Math. Soc. **317** (1990), 585-609

- [4] T.-R. Lin, *Hermitian-Yang-Mills metrics and stability for holomorphic vector bundles with Higgs fields*. Preprint
- [5] G. Lusztig, *On quiver varieties*. Adv. Math. **136** (1998), 141-82
- [6] M. Maruyama, *On boundedness of families of torsion free sheaves*. J. Math. Kyoto Univ. **21** (1981), 673-701
- [7] N. Nitsure, *Moduli space of semistable pairs on a curve*. Proc. London Math. Soc. (3) **62** (1991), 275-300
- [8] Ch. Okonek, A. Schmitt, and A. Teleman, *Master spaces for stable pairs*. Topology **38** (1999), 117-139
- [9] A. Schmitt, *Projective moduli for Hitchin pairs*. Int. J. Math. **9** (1998), 107-118; Erratum **11** (2000), 589.
- [10] M.-S. Stupariu, *The Kobayashi-Hitchin Correspondence for Vortex-Type Equations Coupled with Higgs Fields*. Ph.D.-Thesis, Zürich, 1998
- [11] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*. Invent. Math. **117** (1994), 317-53
- [12] K. Yokogawa, *Moduli for stable pairs*. J. Math. Kyoto Univ. **31** (1991), 311-27
- [13] K. Yokogawa, *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves*. J. Math. Kyoto Univ. **33** (1993), 451-504

*Bar-Ilan University  
Department of Mathematics  
and Computer Science  
Ramat-Gan, 52900  
Israel  
and*

*Universität GH Essen  
FB 6 Mathematik und Informatik  
D-45117 Essen  
Deutschland*

*E-mail address: schmitt1@cs.biu.ac.il*